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Secţia I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

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REMARQUES SUR UNE CONSTRUCTION D'APPLICATIONS À PARTIR D'UNE APPLICATION MULTIVOQUE

PAR

AL. CLIMESCU

1. En nous conformant à un usage plus ancien, nous considérons que la notion de fonction est plus générale que celle d'application univoque : par exemple, on peut concevoir une fonction comme un élément de l'ensemble des parties d'un produit cartésien, ou attribuer la même dénomination aux foncteurs et même aux morphismes.

Le but réel de la préférence accordée au terme fonction est d'arriver à une définition assez générale et satisfaisante de l'équation fonctionnelle, notion dont la complexité est telle qu'elle est plutôt intuitivement perçue que vraiment bien définie sauf pour certaines classes particulières de fonctions. Naturellement il faut commencer par passer en revue les définitions connues de la notion de fonction et au besoin introduire de nouvelles définitions ; il semble d'ailleurs que la plupart des définitions existantes emploient les applications comme élément composant de la définition.

2. Dans cette Note il ne sera question que de ces fonctions particulières, les applications (univoques ou multivoques).

La présentation d'une certaine manière de l'extraction intuitive d'une application univoque à partir d'une application multivoque permet la généralisation immédiate de cette construction et de même son application ultérieure. Mais si l'on cherche à limiter des choix d'abord libres par des conditions convenables — ce qu'on fait couramment en théorie des fonctions d'une variable complexe quand on définit les branches d'une fonction multiforme — des difficultés se présentent, qui seront mises en lumière par quelques exemples appropriés.

3. Soit F une famille de parties A_i d'un ensemble donné. Choisissons pour chaque indice i une fonction de choix a_i pour A_i ce qui permet ensuite de former la famille $\{A_i, a_i\}$ des couples mis en évidence.

Cela permet de présenter systématiquement l'extraction d'une application univoque par le procédé suivant.

Soit X l'ensemble des valeurs de la fonction multivoque e_x et a_x la fonction de choix de X. Une application univoque tirée de f sera par définition le couple

$$< f, |X, a_x| >$$

où $|X, a_x|$ désigne la famille de tous les couples $\{X, a_x\}$ possibles.

En faisant varier les a_x on obtient des familles différentes et en laissant de côté certains des X, on obtient une famille partielle. Alors, *une famille d'applications univoques tirées de l'application multivoque f sera par définition le couple $< f, F >$, où F est un ensemble de familles de couples.*

On remarquera que nous avons tacitement supposé le domaine et le codomaine et que f est partout définie ; pourtant si une famille est partielle, l'application univoque correspondante n'est par partout définie (on dira alors qu'elle est localement définie ; en effet, si $\{X, a_x\}$ est absent de la famille, l'application univoque correspondante n'est pas définie en x.

Pour les applications à plusieurs variables on peut construire des familles d'applications univoques à un nombre moindre de variables en deux étapes :

Soit $f(x_1, x_2,..., x_n)$ une application même univoque à n variables et $p<n$. En fixant $x_1, x_2,..., x_p$ et en laissant libres les autres variables on obtient un ensemble de valeurs qui peut être considéré comme l'expression d'une application $g(x_1, x_2,..., x_p)$ en général multivoque (I-ère étape).

En seconde étape on applique le procédé indiqué ci-dessus.

4. Donnons quelques exemples.

a) Soit $f(x, y) = \{x, y\}$ et en particulier $f(x, x) = \{x\}$, $x, y \in M$.

Soit $A \subset M$ et définissons la fonction de choix de $\{x, y\}$ par $f(x, y) = x$ qui est un choix obligé et

$$f(x, y) = y \text{ si } y \in A \quad (x \neq y), \quad f(x, y) = x \text{ si } y \notin A.$$

On arrive à une application multivoque $g(x)$ en supposant $A \neq \emptyset, M$ et en posant $g(x) = x \in A$.

g on pourra par exemple extraire par les choix

$$g(x) = x \text{ si } x \in A, \quad g(x) = a \text{ si } x \notin A,$$

a étant un élément fixe de A ; en faisant varier a on obtient une famille d'applications univoques.

b) Soit $f(x, y) = \{x, y\}$ et $f(x, y) = x$.

Pour x, y, z donnés différents et pris dans cet ordre définissons les choix de $\{x, y\}$, $\{y, z\}$, $\{x, z\}$ de telle sorte qu'on ait pour l'application locale obtenue (notée aussi f)

$$f(x, f(y, z)) = f(f(x, y)z) ;$$

on a les 6 familles de choix suivantes :

1. $f(x, y) = x, \quad f(y, z) = y, \quad f(x, z) = x$;
2. $f(x, y) = x, \quad f(y, z) = z, \quad f(x, z) = x$;
3. $f(x, y) = x, \quad f(y, z) = z, \quad f(x, z) = z$;

4. $f(x, y) = y, \quad f(y, z) = y, \quad f(x, z) = x$;

5. $f(x, y) = y, \quad f(y, z) = y, \quad f(x, z) = z$;

6. $f(x, y) = y, \quad f(y, z) = z, \quad f(x, z) = z$;

les deux autres choix possibles

7. $f(x, y) = x, \quad f(y, z) = y, \quad f(x, z) = z$;

8. $f(x, y) = y, \quad f(y, z) = z, \quad f(x, z) = x$,

corespondant à la condition

$$f(x, f(y, z) \neq f(f(x, y), z).$$

En ajoutant le choix obligatoire $f(x, y) = x$, on obtient une famille d'applications localement définies et on peut se demander si l'on ne peut faire des choix concordants c'est-à-dire simultanément compatibles et qui pris tous à la fois détermineraient une seule application univoque partout définie. On peut montrer pour le second cas (choix 7 et 8) que cela n'est même pas possible pour trois éléments et leurs huit ordres possibles.

c). Dans cet exemple il ne sera question que de nombres réels. Pour $a \leqslant x < y \leqslant b$ définissons la fonction multivoque f par

$$f(x, y) = \{1, -1\}.$$

Afin de faire le choix acceptons la règle suivante.

Si $a = a_0 < a_1 < a_2 \ldots < a_n = b$ est une partition de l'intervalle et si $x = a_p$, $y = a_{p+1}$, prenons $f(x, y) = (-1)^p$. Chaque partition donne naissance à une application localement définie et une famille de partitions donnera une famille d'applications localement définies.

Pour cet exemple, on pourrait songer aussi à des choix aléatoires expérimentalement déterminés.

5. Ces exemples qu'on pourrait aisément multiplier, montrent que la variété des conditions est telle que leur formalisation adéquate est très difficile et dépasse peut-être la force des systèmes formels usuels.

Dans le cas de l'exemple b) il est intuitivement désirable de considérer f comme une application partout définie et conçue comme univoque, bien que pour chaque paire (x, y) on doive encore indiquer la fonction de choix utilisée. Cela semble indiquer qu'on serait obligé d'accepter comme existante une fonction univoque qui ne vérifierait pas le principe d'identité. Bien entendu si l'on recule devant cette éventualité il reste la possibilité de revenir à la famille d'applications localement définies, d'autant plus que l'abandon du principe d'identité change considérablement la logique elle-même.

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REMARCI ASUPRA UNEI CONSTRUCŢII DE APLICAŢII UNIVOCE PLECÎND DE LA O APLICAŢIE MULTIVOCĂ

(Rezumat)

Prin generalizarea unui procedeu intuitiv de extragere a unei aplicaţii univoce se obţine o familie de aplicaţii univoce extrase (eventual local adică nu peste tot definite). Se dau de asemenea trei exemple dintre care unul conduce intuitiv la problema de existenţă a unei aplicaţii care n-ar verifica principiul identităţii.

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A NORMED SPACE OF POLYGENIC FUNCTIONS

BY

JOHN SYNOWIEC

1. Introduction

Non-analytic complex functions have been studied by a number of mathematicians, including Kasner [6], Bers [1], and Vekua [9]. The purpose of this paper is to study the polygenic functions of Kasner [6], [3], by means of a topology constructed from the Kasner radius. Such a topology is of interest because the Kasner radius measures the deviation of a function from analyticity.

The natural way of using the Kasner radius to form a neighborhood system is described in § 2. This leads to the introduction of equivalence classes of polygenic functions, which form a vector space. Theorem 2.5 states that this vector space X is a topological group with invariant metric. However, scalar multiplication is not continuous, so that X is not a topological vector space. In § 3, a boundedness requirement is added for the polygenic functions, and the result is a subspace P of X which is not only a topological vector space, but is also normable. (Theorem 3.3). In § 4, various properties of the space P are considered.

2. A Topological Group of Polygenic Functions

Let R be an open subset of the complex plane, and let X_0 denote the set of all polygenic functions with continuous first order derivatives on R. For f in X_0, and for $\varepsilon > 0$, let an ε-neighborhood, $N(f,\varepsilon)$, be defined by

$$N(f, \varepsilon) = \left\{ g \in X_0 \,\middle|\, \left| \frac{\partial}{\partial z}(f - g) \right| < \varepsilon \text{ for all } z \in R \right\},$$

and let

$$B(f) = \{N(f, \varepsilon) \mid \varepsilon > 0\}, \text{ for each } f \text{ in } X_0.$$

Proposition 2.1. i) *For each* $N(f, \varepsilon) \in B(f)$, $f \in N(f, \varepsilon)$;

ii) *If* N_1, $N_2 \in B(f)$, *then there is an* $N \in B(f)$ *with* $N \subseteq N_1 \cap N_2$;

iii) *If* $N \in B(f)$, *then there is an* $N' \in B(f)$ *such that for each* $g \in N'$, N *contains a set in* $B(g)$.

Properties i) and ii) are clear. To show iii), choose $N(f, \varepsilon) \in B(f)$, and let $\eta \in (0, \varepsilon)$. Let $N' = N(f, \eta)$, and let $\xi \leqslant \varepsilon - \eta$. Then if $g \in N(f, \eta)$, and $g' \in N(g, \xi)$,

$$\left| \frac{\partial}{\partial \bar{z}} (g' - f) \right| < \xi + \eta \leqslant \varepsilon.$$

By Proposition 2.1, the set $B(f)$ is a neighborhood base at f, i. e.,a set $N \subseteq X_0$ is a neighborhood of f if there is some $N(f, \varepsilon) \in B(f)$ such that $N(f, \varepsilon) \subseteq N$. Thus in the unique topology on X_0 determined by $B(f)$, a set $U \subseteq X_0$ is open if for each $f \in U$, there is an N in $B(f)$ such that $N \subseteq U$.

The set of all holomorphic functions on R will be denoted by $H(R)$, or by H. Thus, $h \in H$ if and only if $h \in X_0$ and $\partial h / \partial \bar{z} = 0$, on R.

The topology on X_0 determined by the base

$$B = \bigcup_{f \in X_0} B(f),$$

is evidently not a Hausdorff space, since for any $f \in X_0$, $f + h$ and f belong to all neighborhoods of f, if h is any holomorphic function on R. Also, it suffices to take rational values of ε, in order that the sets $N(f, \varepsilon)$ form a neighborhood base at f, i. e., X_0 is a first countable space.

Note that for each $f \in X_0$ and each $\varepsilon > 0$, $N(f, \varepsilon)$ is an uncountable set. For, if α is any real number such that $0 \leqslant \alpha < \varepsilon$, then $f + \alpha \bar{z} \in N(f, \varepsilon)$. Moreover, any $h \in H$ can be added to such a function: $f(z) + h(z) + \alpha \bar{z} \in N(f, \varepsilon)$.

Proposition 2.2. X_0 *is a complex vector space, and* H *is a vector subspace of* X_0.

Since X_0 is not a Hausdorff space, we introduce the following equivalence relation on X_0, $f \sim g$ if and only if $f - g \in H$.

The resulting quotient space will be denoted by $X = X_0 / H$.

If $[f]$ denotes the equivalence class determined by f, then the natural mapping $\varphi : X_0 \to X_0 / H$, is defined by $\varphi(f) = [f]$. If addition and scalar multiplication are defined by $[f] + [g] = [f + g]$, and $\alpha[f] = [\alpha f]$, then $X = X_0 / H$, is a complex vector space in which the null vector consists of all holomorphic functions on R. If $g_1, g_2 \in [f]$, then $\partial g_1 / \partial \bar{z} = \partial g_2 / \partial \bar{z}$, on R, since $g_1 = f + h_1$, $g_2 = f + h_2$, where $h_1, h_2 \in H$.

A fundamental system of neighborhoods of $[f] \in X$, is given by:

$$N([f], \varepsilon) = \left\{ [g] \in X \,\middle|\, \left| \frac{\partial}{\partial \bar{z}} (f - g) \right| < \varepsilon \text{ for all } z \in R \right\}.$$

Proposition 2.3. *The topology generated by the sets $N([f], \varepsilon)$ as $[f]$ varies over X and ε varies over all positive rational numbers is a Hausdorff topology.*

For, if $[g]$ is in the intersection of all the sets $N([f], \varepsilon)$, where $\varepsilon > 0$, then for each $\varepsilon > 0$,

$$\left| \frac{\partial}{\partial \bar{z}} (g - f) \right| < \varepsilon \text{ for all } \varepsilon > 0,$$

so that $g - f \in H$, i. e., $[g] = [f]$. Hence, if $[g] \neq [f]$, there is a neighborhood $N([f], \varepsilon_1)$ for which $[g] \notin N([f], \varepsilon_1)$, and there is a neighborhood $N([g], \varepsilon_2)$ such that $[f]$ is not in $N([g], \varepsilon_2)$. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then

$$\left| \frac{\partial}{\partial \bar{z}} (g - f) \right| \geqslant \varepsilon_1 \geqslant \varepsilon, \text{ and } \left| \frac{\partial}{\partial \bar{z}} (f - g) \right| \geqslant \varepsilon_2 \geqslant \varepsilon.$$

Thus, if $[h] \in N\left([f], \frac{\varepsilon}{2}\right) \cap N\left([g], \frac{\varepsilon}{2}\right)$,

$$\left| \frac{\partial}{\partial \bar{z}} (f - g) \right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is impossible, so that $N([f], \varepsilon/2)$ and $N([g], \varepsilon/2)$ are disjoint.

Evidently, X is also a first countable space.

Theorem 2.4. *X is a commutative topological group with respect to addition of functions.*

If W is a neighborhood of $[f_0] + [g_0] \in X$, then there is some $N([f_0 + g_0], \varepsilon) \subseteq W$. Thus, if $[f] \in N([f_0], \varepsilon/2)$, and $[g] \in N([g_0], \varepsilon/2)$, then

$$\left| \frac{\partial}{\partial \bar{z}} (f + g - f - g) \right| \leqslant \left| \frac{\partial}{\partial \bar{z}} (f - f_0) \right| + \left| \frac{\partial}{\partial \bar{z}} (g - g_0) \right| < \varepsilon,$$

that is, $N([f_0], \varepsilon/2) + N([g_0], \varepsilon/2) \subseteq N([f_0 + g_0], \varepsilon) \subseteq W$.

Also, if V is any neighborhood of $-[f_0]$, there is some $N(-[f_0], \varepsilon)$ contained in V. Then $N([f_0], \varepsilon) \subseteq -V$, so that V is also a neighborhood of $[f_0]$.

This shows that addition and the operation of forming inverses are continuous.

In [5, p. 70], the following result is established. A T_0-topological group is metrizable if and only if there exists a countable open base at the iden-

tity e, and such a metric can be taken to be left invariant. Using this result and Propostion 2.3, Theorem 2.4, we find:

Theorem 2.5. *X is a metrizable commutative topological group with invariant metric.*

The proof of the following theorem is straighforward.

Theorem 2.6. *The family $B(0)$ of neighborhoods of zero satisfies the following four conditions:*

i) *Each N in $B(0)$ is balanced;*

ii) *If $N \in B(0)$, and α is any non-zero complex number, then $\alpha N \in B(0)$;*

iii) *If $N \in B(0)$, there is an $N' \in B(0)$ such that $N' + N' \subseteq N$;*

iv) *If N, $N' \in B(0)$, there is an $N'' \in B(0)$ such that $N'' \subseteq N \cap N'$.*

If each neighborhood N of zero were also absorbant, i. e., for each $[f] \in X$, $\alpha > 0$, exists such that $\lambda[f] \in N$, for all complex numbers λ with $|\lambda| \leqslant \alpha$, then X would be a topological vector space [4, pp. 56—57]. Since this condition would imply that for each $[f] \in X$, $\alpha > 0$, would exist with

$$\left|\frac{\partial f}{\partial \bar{z}}\right| < \frac{\varepsilon}{|\lambda|}, \quad |\lambda| \leqslant \alpha,$$

then each neighborhood of zero in X is not absorbing. It follows that scalar multiplication is not continuous.

Thus X is an example of a Hausdorff topological space which is also a vector space, but is not a topological vector space, although it is a topological group.

3. A Normed Vector Space of Polygenic Functions

Since the topology of X is not compatible with its vector space structure, we consider the subset

$$P = \left\{ [f] \in X \,\middle|\, \underset{R}{\text{l.u.b.}} \left|\frac{\partial f}{\partial \bar{z}}\right| < \infty \right\},$$

of X, with the induced topology. Evidently, P is a vector subspace of X.

Theorem 3.1. *P is a locally convex Hausdorff topological vector space with respect to the topology induced by X.*

The properties listed in Theorem 2.6 are also true for P, and the topology is Hausdorff.

It is seen that each neighborhood of zero in P is absorbing: If $N \in B(0)$, and $[f] \in P$, choose

$$\alpha = \frac{1}{2} \frac{\varepsilon}{\underset{R}{\text{l.u.b.}} \left|\frac{\partial f}{\partial \bar{z}}\right|}.$$

Then if $|\lambda| \leqslant \alpha$,

$$\left|\frac{\partial}{\partial \bar{z}}(\lambda f)\right| = |\lambda| \left|\frac{\partial f}{\partial \bar{z}}\right| < \varepsilon.$$

Finally, each neighborhood $N(0, \varepsilon)$, of zero is convex.

Kolmogorov's normability criterion states that a Hausdorff topological vector space X is normable if and only if there is at least one bounded, open, convex set in X. ([4, p. 85] or [8, pp. 135—137]).

Theorem 3.2. *P is a normable topological vector space.*

Clearly each $N(0, \varepsilon)$ is open and convex. If U is a neighborhood of zero in P, there exists some $N(0, \eta) \subseteq U$. Choose $\alpha > 0$ such that $\alpha > \varepsilon/\eta$. Then $\varepsilon < \alpha\eta$ implies $N(0, \varepsilon) \subseteq N(0, \alpha\eta) = \alpha N(0, \eta) \subseteq U$, i. e., U is a bounded set.

The neighborhood $N(0, \varepsilon)$ of 0 is balanced, convex, and bounded. Then the Minkowski functional p of $N(0, \varepsilon)$, is

$$p([f]) = \text{g. l. b.} \{ \alpha > 0 \mid [f]/\alpha \in N(0, \varepsilon)\}.$$

Since $[f]/\alpha \in N(0, \varepsilon)$ implies

$$\frac{1}{\alpha}\left|\frac{\partial f}{\partial \bar{z}}\right| < \varepsilon,$$

so that

$$p([f]) = \frac{1}{\varepsilon} \underset{R}{\text{l. u. b.}} \left|\frac{\partial f}{\partial \bar{z}}\right| < \infty.$$

If we take $\varepsilon = 1$, we obtain the norm

$$\| [f] \| = \underset{R}{\text{l.u.b.}} \left|\frac{\partial f}{\partial \bar{z}}\right|.$$

Remarks. If R is unbounded, then P cannot contain the functions $\bar{z}^2, \bar{z}^3, \ldots, \bar{z}^n, \ldots$. In particular, P cannot contain the analytic polygenic functions of P. Krajkiewicz.

Bers [2] defines a pseudoanalytic function of the first kind on a bounded domain R to be a complex-valued function f with continuous derivatives of first order which satisfies an equation of the form

$$\frac{\partial f}{\partial \bar{z}} = af + b\bar{f},$$

where a, b, are bounded, Holder continuous complex-valued functions defined on R.

In the case where R is the unit circle with center at the origin, the function $f(z) = \bar{z}$, satisfies $[f] \in P$, since $\partial f/\partial \bar{z} = 1$. However, if f were to be pseudoanalytic, then we would have $1 = a\bar{z} + bz$, and if $z = 0$, then this equation is impossible. Thus, f is not pseudoanalytic of the first kind.

Evidently, if f is a bounded pseudoanalytic function:

$$|f(z)| \leqslant M, \text{ for all } z \text{ in } R,$$

then

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leqslant (|a| + |b|)M,$$

i. e., $[f] \in P$. Clearly, the bounded pseudoanalytic functions form a vector subspace of P.

4. Some Properties of the Space P

In this section, we discuss the connection between completeness of P and generalized differentiation.

Let P_0 denote the set of all polygenic functions f on R such that

$$p(f) = \underset{R}{\text{l.u.b.}} \left\{ \left|\frac{\partial f}{\partial \bar{z}}\right| \right\} < \infty.$$

Then P_0 is a complex vector space, and p is a semi-norm on P_0.

Theorem 4.1. *Let (f_n) be a sequence of polygenic functions in P_0 such that (f_n) is a Cauchy sequence with respect to the semi-norm p. Then there is a bounded continuous function g on R such that*

$$\text{l.u.b.} \left|\frac{\partial f_n}{\partial \bar{z}} - g\right| \to 0, \quad \text{as} \quad n \to \infty.$$

Since (f_n) is Cauchy with respect to p,

$$\lim_{m,\, n \to \infty} \underset{R}{\text{l.u.b.}} \left|\frac{\partial}{\partial \bar{z}}(f_n - f_m)\right| = 0,$$

so that $(\partial f_n/\partial z)$ converges uniformly on R to a function g, which must be continuous. Moreover, since each function of this sequence is bounded, the function g must be bounded.

In general, the function g of Theorem 4.1 is not polygenic. For example, let R be the open unit square in the complex plane, and let w be a continuous nowhere differentiable function on $[0, 1]$. Let u be the indefinite integral of w from 0 to any x in $[0, 1]$, and define $f(x, y) = u(x) + u(y)$.

Then the partial derivatives $\partial f/\partial x = u(x)$, $\partial f/\partial y = u(y)$, are continuous on the closure of R, and hence are bounded on R. Thus f is a polygenic function on R, and

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}[u(x) + \mathrm{i}\, u(y)] = g(x, y),$$

so that $\partial f/\partial \bar{z}$ is bounded on R, i. e., $f \in P_0$, but $g \notin P_0$.If (f_n) is the sequence defined by $f_n = f$, for each n, then Theorem 4.1 is satisfied, and the limit $g \notin P_0$.

Passing to the quotient space, P, we have

$$\| [f_n] - [f_m] \| = \| [f_n - f_m] \| = p(f_n - f_m) < \varepsilon,$$

for m, n, sufficiently large, so that $([f_n])$ is a Cauchy sequence in P, with no limit in P.

Thus, P is not a complete normed vector space.

In [7], generalized polygenic functions on a bounded connected open set R of the complex plane were introduced by means of Friedrichs-Sobolev derivatives: if $f \in L^1(R)$, then $D_{\bar{z}} f$ is a function defined by:

$$\iint_R D_{\bar{z}} f \varphi \, \mathrm{d}A = -\iint_R f \frac{\partial \varphi}{\partial \bar{z}} \mathrm{d}A,$$

for all polygenic functions φ with compact support in R.

Let

$$D_1(R) = \{ f \in L'(R) \mid D_{\bar{z}} f \in L^1(R) \},$$

and let $p_1(f)$ denote the L-norm $\| D_{\bar{z}} f \|_1$, of f.

Let $C(R)$ denote the space of all bounded continuous complex-valued functions on R, with the sup-norm. Define a map $T : P_0(R) \to C(R)$, by $T(f) = \partial f/\partial \bar{z}$, for each $f \in P_0(R)$.

Then $T(P_0)$ is a proper vector subspace of $C(R)$, and T is extended to $D_1(R)$ by the

Theorem 4.2. *If (f_n) is a sequence in $P_0(R)$ which is Cauchy with respect to the semi-norm p, there is a bounded continuous function $g \in C(R)$, such that*

$$\lim_{n \to \infty} \left\| \frac{\partial f_n}{\partial \bar{z}} - g \right\| = 0,$$

where g is the Friedrichs-Sobolev derivative of a function $f \in D_1(R)$, and

$$\lim_{n \to \infty} \left\| \frac{\partial f_n}{\partial \bar{z}} - D_{\bar{z}} f \right\|_1 = 0.$$

For, g is clearly in $L^1(R)$, and if

$$f(z) = -\frac{1}{\pi} \iint_R \frac{g(\xi, \eta)}{\zeta - z} \mathrm{d}\xi \mathrm{d}\eta,$$

then $f \in L^1(R)$, and $D_{\bar{z}} f = g$. [9, p. 34]. Thus

$$\left\| D_{\bar z} f_n - g \right\|_1 = \left\| \frac{\partial f_n}{\partial \bar z} - g \right\|_1 = \iint_R \left| \frac{\partial f_n}{\partial \bar z} - g \right| \mathrm{d}A \leqslant \left\| \frac{\partial f_n}{\partial \bar z} - g \right\| m(R),$$

where $m(R)$ is the measure of R.

Corollary 1. *Let $P_1 = D_1(R)/H$. If $([f_n])$ is any Cauchy sequence in P, there is some $[f] \in P_1(R)$ such that*

$$\lim_{n\to\infty} \left\| \frac{\partial f_n}{\partial \bar z} - g \right\| = 0, \quad \text{and} \quad [D_{\bar z} f] = [g],$$

and $\| [D_{\bar z} f_n] - [g] \|_1 \to 0$, *as* $n \to \infty$.

Let BP_0 denote the subspace of P_0 consisting of bounded polygenic functions, and let BP denote the corresponding quotient space $BP = BP_0/H$.

Corollary 2. *The completion of BP is a closed subspace of $P_1(R)$.*

Note that BP_0 is a subspace of $D_1(R)$.

Suppose $P_0(R)$ is replaced by the space $P_0(R, \alpha)$ of all continuous complex-valued functions f such that $\partial f/\partial \bar z$ is locally Holder continuous on R. That is, for some fixed α, with $0 < \alpha < 1$, each point z of R is contained in an open sphere $S(z, r) \in R$, about z, such that if z_1, z_2 are any two points of $S(z, r)$, then

$$\left| \frac{\partial f}{\partial \bar z}(z_1) - \frac{\partial f}{\partial \bar z}(z_2) \right| \leqslant K \, | z_1 - z_2 |^{\alpha}, \text{ for some } K > 0.$$

In this case, the quotient space $P(R, \alpha) = P_0(R, \alpha)/H$, should be a Banach space, so that ordinary derivatives are sufficient in this case.

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UN SPAŢIU NORMAT DE FUNCŢII POLIGENICE

(Rezumat)

Se studiază funcţiile poligenice ale lui K a s n e r [6], [3], utilizînd o topologie construită pornind de la raza lui Kasner. Se introduc clase de echivalenţă de funcţii poligenice, care formează un spaţiu normat şi se demonstrează că acest spaţiu este grup topologic cu invariant metric, fără a fi spaţiu vectorial topologic. Cu anumite restricţii pentru funcţiile poligenice, spaţiul devine nu numai vectorial topologic, ci şi normabil.

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FIBRATIONS IN THE HOMOTOPICAL SENSE [1])

BY

I. POP

1. In this paper is given a definition of fibrations in the homotopical sense (f. h. s) (definitions 2, 3).

Theorem 1 proves that a map p is a f. h. s. if and only if there exists a „lifting couple" for p (theorem 1). The dual concept of f. h. s.-cofibrations in the homotopical sense is defined and it is proved that a map is a cofibration in the homotopical sense (c. h. s) if and only if there exists a „retracting function" in the homotopical sense for f (theorem 2). A construction of f. h. s from c. h. s is also proposed (theorem 3).

2. Let $p: E \to B$ and $f: X \to B$ be two continuous maps. We remember :

D e f i n i t i o n 1. The map f has a lifting in the homotopical sense with respect to p if there exists a map $f': X \to E$ such that $pf' \simeq f$. Let be the diagram

$$\begin{tikzcd} E \arrow[r, "p"] & B \\ X \arrow[u, "f"] & X \times I \arrow[u, "F"'] \end{tikzcd}$$

n which F is a homotopy for pf, therefore $F(x, 0) = pf(x)$.

D e f i n i t i o n 2. We say that the map p has the homotopy lifting property in the homotopical sense, with respect to the space X, if there exists a continuous map $F': X \times I \to E$ such that $F'(x, 0) = f(x)$ and $pF' \simeq \simeq F \operatorname{rel}(X \times \dot{I})$.

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[1]) Presented at the scientific session „Al. Myller" of the University „Al. I. Cuza", Jassy, August 20–25, 1970.

Definition 3. A continuous map $p : E \to B$ is called a *fibration in the homotopical sense* (f. h. s) if it has the homotopy lifting property in the homotopical sense with respect to any topological space X.

We present some properties of the f. h. s whose proofs are direct.

Proposition 1. *If $p : E \to B$ is a f. h. s. then for every path $\omega \in B^I$ with $\omega(0) = p(e)$ there exists a path $\omega' \in E^I$ with $\omega'(0) = e$ and $p\omega' \simeq \omega$ rel $\dot{I}$.*

Proposition 2. *Let $p : E \to B$ a f. h. s. and X be a locally compact Hausdorff space. If we define the map $p' : E^X \to B^X$ by $p'(g) = p \circ g$ for $g \in E^X$ then p' is also a f. h. s.*

Proposition 3. *The product and composition of f. h. s are f. h. s.*

Proposition 4. *Let $p : E \to B$ a f. h. s. and $F_0, F_1 : X \times I \to E$ be two continuous maps. Given the homotopy $H : p \circ F_0 \simeq pF_1$ and $G : F_0/X \times 0 \simeq F_1/X \times 0$ such that $H(x, 0, t) = pG(x, 0, t)$, there exists the homotopy $H' : F_0 \simeq F_1$ which satisfies $pH' \simeq H$ rel $(X \times I \times \dot{I})$.*

Corollary. Let $p : E \to B$ a f. h. s. and $F_0, F_1 : X \times I \to E$ be liftings of the same map such that $F_0/X \times 0 = F_1/X \times 0$; then $F_0 \simeq F_1$ rel $(X \times \dot{I})$.

We give now a necessary and sufficient condition such that a map be a f. h. s.

For this, let $p : E \to B$ be a continuous map. We consider the space $\bar{B} = \{(e, \omega) \in E \times B^I / \omega(0) = p(e)\}$ with the subspace topology (B^I with compact open topology and $E \times B^I$ with product topology).

Definition 4. Let $\lambda : \bar{B} \to E^I$ and $h : \bar{B} \to B^{I \times I}$ be two continuous maps. We say that the couple (λ, h) is a *lifting couple* for p if it satisfies the conditions:

1. $\lambda(e, \omega)(0) = e$,
2. $h(e, \omega) : p \circ \lambda(e, \omega) \simeq \omega$ rel $\dot{I}$.

Theorem 1. *A continuous map $p : E \to B$ is a f. h. s if and only if there exists a lifting couple for p.*

Proof. Let $p : E \to B$ be a f. h. s. Let $f' : \bar{B} \to E$ and $F : \bar{B} \times I \to B$ be two maps defined by $f'(e, \omega) = e$ and $F((e, \omega), t) = \omega(t)$. We have $F((e, \omega), 0) = p(e) = pf'(e, \omega)$ and, since p is a f. h. s., there exists the map $F' : \bar{B} \times I \to E$ such that $F'((e, \omega), 0) = f'(e, \omega) = e$ and $pF' \simeq F$ rel $(\bar{B} \times \dot{I})$.

We define the map $\lambda : \bar{B} \to E^I$ by $\lambda(e, \omega)(t) = F'((e, \omega), t)$. We have $\lambda(e, \omega)(0) = F'((e, \omega), 0) = e$ and $p \circ \lambda(e, \omega)(1) = pf'((e, \omega), 1) = F((e, \omega), 1) = \omega(1)$.

If $H : (\bar{B} \times I) \times I \to B$ is a homotopy with:

$H((e, \omega), t, 0) = pF'((e, \omega), t)$,

$H((e, \omega), t, 1) = F((e, \omega), t) = \omega(t)$,

$H((e, \omega), 0, \tau) = p(e)$ and $H((e, \omega), 1, \tau) = \omega(1)$,
then we define the map $h : \bar{B} \to B^{I \times I}$ by $h(e, \omega)(t, \tau) = H((e, \omega), t, \tau)$. We have: $h(e, \omega)(t, 0) = p\lambda(e, \omega)(t)$, $h(e, \omega)(t, 1) = \omega(t)$, $h(e, \omega)(0, \tau) = \omega(0)$ and $h(e, \omega)(1, \tau) = \omega(1)$, therefore $h(e, \omega) : p \circ \lambda(e, \omega) \simeq \omega$ rel $\dot{I}$ that is (λ, h) is a lifting couple for p.

We prove now the inverse of this theorem. Let $p : E \to B$ a continuous map and (λ, h) be a lifting couple for p. Let be the diagram

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ f \uparrow & & \uparrow F \\ X & & X \times I \end{array} \quad \text{with } pf(x) = F(x, 0).$$

We define the map $g : X \to B^I$ with $g(x)(t) = F(x, t)$. Then $(f(x), g(x)) \in \bar{B}$ and therefore we can define the map $F' : X \times I \to E$ by $F'(x, t) = \lambda(f(x), g(x))(t)$.

We have $F'(x, 0) = \lambda(f(x), g(x))(0) = f(x)$. We prove that $pF' \simeq F$ rel $(X \times \dot{I})$. For this, let be the map $H' : (X \times I) \times I \to E$ defined by $H'((x,t),\tau) = h(f(x), g(x))(t, \tau)$. We have directly $H' : pF' \simeq F$ rel $(X \times \dot{I})$.

Observation. If the map p has a lifting function $\lambda : \bar{B} \to E^I$ and if we consider $h(e, \omega)(t, \tau) = \omega(t)$ then the couple (λ, h) is a lifting couple for p. Therefore every H-fibration is a f. h. s. Let $p : E \to B$ a continuous map and W be a subspace of B^I. We consider the space $\widetilde{W} = \{(e, \omega, s) \in E \times W \times I / \omega(s) = p(e)\}$.

Definition 5. If $\Lambda : \widetilde{W} \to E^I$ and $H : \widetilde{W} \to B^{I \times I}$ are two continuous maps, then the couple (Λ, H) is called an extended lifting couple over W if we have:

1') $\Lambda(e, \omega, s)(s) = e$,

2') $H(e, \omega, s) : p \circ \Lambda(e, \omega, s) \simeq \omega$ rel $(\{s\} \cup \{1 - s\})$.

Lemma. *If the map $p : E \to B$ has an extended lifting couple over B^I then p is a f. h. s.*

Proof. Let (Λ, H) be an extended lifting couple over B^I for p. Therefore we have $\Lambda(e, \omega, 0)(0) = e$ and $H(e, \omega, 0) : p \circ \Lambda(e, \omega, 0) \simeq \omega$ rel $\dot{I}$. If we define the map $\lambda : \bar{B} \to E^I$ by $\lambda(e, \omega) = \Lambda(e, \omega, 0)$ and the map $h : \bar{B} \to B^{I \times I}$ by $h(e, \omega) = H(e, \omega, 0)$ then (λ, h) is a lifting couple for p and in view of theorem 1, p is a f. h. s.

I don't know if the inverse theorem is true.

3. Now, we consider the dual concep of f. h. s.

Definition 6. A map $f : X' \to X$ is called a cofibration in the homotopical sense (c. h. s) if for every continuous map $g : X \to Y$ and

for $G: X' \times I \to Y$ with $G(x' 0) = g(f(x))$ there exists a continuous map $F: X \times I \to Y$ such that $F(x, 0) = g(x)$ and $F \circ (f \times 1) \simeq G \operatorname{rel} (X' \times \dot{I})$.

Let $\bar{\bar{X}}$ be the quotient space of the sum $(X' \times I) \vee (X \times 0)$ obtained by identifying $(x', 0) \in X' \times I$ with $(f(x'), 0) \in X \times 0$ for all $x' \in X'$. We use $[x', t]$ and $[x, 0]$ to denote the points of $\bar{\bar{X}}$ coresponding to $(x', t) \in X' \times I$ and $(x, 0) \in X \times 0$, respectively. Then $[x', 0] = [f(x'), 0]$. There exists a map $\bar{i}: \bar{\bar{X}} \to X \times I$ defined by $\bar{i}[x', t] = (f(x'), t)$ and $\bar{i}[x, 0] = (x, 0)$. Also we can define the map $j: X' \times I \to \bar{\bar{X}}$ by $j(x', t) = [x', t]$.

Definition 7. A retracting function in the homotopical sense for f is called a continuous map $\rho: X \times I \to \bar{\bar{X}}$ such that $\rho(x, 0) = [x, 0]$ and $\rho \bar{i} \simeq 1 \operatorname{rel}(j(X' \times \dot{I}))$.

Theorem 2. *A continuous map $f: X' \to X$ is a c. h. s. if and only if there exists a retracting function in the homotopical sense for f.*

Proof. Let $f: X' \to X$ be a c. h. s. Let $g: X \to \bar{\bar{X}}$ and $G: X' \to \bar{\bar{X}}$ be two maps defined by $g(x) = [x, 0]$ and $G(x', t) = [x', t]$. Since $G(x', 0) = g(f(x'))$ it results that there exists a continuous map $\rho: X \times I \to \bar{\bar{X}}$ such that $\rho(x, 0) = g(x) = [x, 0]$ and $\rho \circ (f \times 1) \simeq G \operatorname{rel}(X' \times \dot{I})$. We have $\rho \circ (f \times 1)(x', t) = \rho \bar{i}[x', t]$.

Let $H: X' \times I \times I \to \bar{\bar{X}}$ be such that $H: \rho \circ (f \times 1) \simeq G \operatorname{rel} (X' \times \dot{I})$. We define the map $h: \bar{\bar{X}} \times I \to \bar{\bar{X}}$ by $h([x', t], \tau) = H(x', t, \tau)$ and $h([x, 0], \tau) = \rho(x, 0)$. Then we have: $h([x', t], 0) = \rho \bar{i}[x', t]$; $h([x', t], 1) = [x', t]$; $h([x', 0] \tau) = [x', 0]$ and $h([x', 1], \tau) = [x', 1]$. Therefore $h: \rho \bar{i} \simeq 1 \operatorname{rel} (j(X' \times \dot{I}))$.

We prove the inverse. Let $g: X \to Y$ and $G: X' \times I \to Y$ be two continuous maps with $G(x', 0) = g(f(x'))$. We define the map $\bar{G}: \bar{\bar{X}} \to Y$ by $\bar{G}([x', t]) = G(x', t)$ and $G([x, 0]) = g(x)$. If $\rho: X \times I \to \bar{\bar{X}}$ is a retracting function in the homotopical sense for f, then we consider the map $F = \bar{G} \circ \rho: X \times I \to Y$ and we have $F(x, 0) = g(x)$. Moreover $F \circ (f \times 1)(x', t) = \bar{G} \rho(f(x'), t) = \bar{G} \rho \bar{i}[x', t]$. Now let $h: \bar{\bar{X}} \times I \to \bar{\bar{X}}$ be such that $h: \rho \bar{i} \simeq 1 \operatorname{rel} (j(X' \times \dot{I})$. We can consider the map $H = \bar{G} \circ h \circ (j \times 1): X' \times X \times I \to Y$ and ve have $H: F \circ (f \times 1) \simeq G \operatorname{rel}(X' \times \dot{I})$, therefore f is a c. h. s.

4. In view of the theorems 1 and 2 we prove a theorem which permit us construction of f. h. s. using c. h. s.

Theorem 3. *Let $f: X' \to X$ be a c. h. s. with X' and X locally compact Hausdorff spaces and Y be any topological space. Then the map $p: Y^X \to Y^{X'}$ defined by $p(g) = g$ is a f. h. s.*

Proof. Let $\rho: X \times I \to \bar{\bar{X}}$ be a retracting function in the homotopical sense for f (theorem 2). Let $\rho': Y^{\bar{\bar{X}}} \to Y^{X \times I}$ be the continuous map defined by $\rho'(g) = g \circ \rho$ for all $g: \bar{\bar{X}} \to Y$. We verify that p is a f. h. s proving the existence of a lifting couple for p (theorem 1). For this let be the

space $\overline{Y^X} = \{(g, G) \in Y^X \times (Y^{X'})^I / g \circ f = G(0)\}$. We define the continuous map $\psi : \overline{Y^X} \to \overline{Y^{\bar{X}}}$ by $\psi(g, G)\,[x', t] = G(t)\,(x')$ and $\psi(g, G)\,[x, 0] = g(x)$ which is well defined.

Now we define the map $\lambda : \overline{Y^X} \to (Y^X)^I$ by $\lambda(g, G)(t)(x) = \rho'(\psi(g, G))\,(x, t)$. Then we have $\lambda(g, G)\,(0)\,(x) = g(x)$ therefore $\lambda(g, G)\,(0) = g$. There exists a map $k : \bar{\bar{X}} \times I \to \bar{\bar{X}}$ such that $k : \bar{\rho} i \simeq 1 \mathrm{rel}\,(j(X' \times I))$. We can consider the continuous map $k' : Y^{\bar{\bar{X}}} \to Y^{\bar{\bar{X}} \times I}$ defined by $k'(g) = g \circ k$ for all $g : \bar{\bar{X}} \to Y$. Finally we can define the map $h : \overline{Y^X} \to (Y^{X'})^{I \times I}$ by $h(g, G)\,(t, \tau)\,(x') = k'\,(\psi(g, G)\,(j(x', t), \tau))$. Then we have $h(g, G)\,(t, 0)\,(x') = p\lambda(g, G)(t)(x')$. Moreover $h(g, G)(t, 1)\,(x') = G(t)\,(x')$. Therefore $h(g, G)$ is a homotopy. Moreover a relative one. In fact $h(g, G)\,(0, \tau)\,(x') = p\lambda(g, G)\,(0)$ and $h(g, G)\,(1, \tau)\,(x') = G(1)\,(x')$, therefore $h(g, G) : p \circ \lambda(g, G) \simeq G \mathrm{ rel } \dot{I}$ that is (λ, h) is a lifting couple for p and therefore it is a f. h. s.

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FIBRĂRI ÎN SENS HOMOTOP

(Rezumat)

Se introduc noţiunile de fibrare în sens homotop (f. h. s) şi cofibrare în sens homotop (c. h. s) şi se dau condiţii necesare şi suficiente ca o aplicaţie continuă să fie f. h. s. şi respectiv c. h. s.

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МАТЕМАТИЧЕСКИЕ СИСТЕМЫ СЛОЖНЫХ СТРУКТУР
I. О РЕШЕНИИ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОБЫКНОВЕННЫМИ И ТЕПЛОВЫМИ ОПЕРАТОРАМИ

Д. МАНЖЕРОН, М. Н. ОГЮЗГОРЕЛИ и Л. Е. КРИВОШЕИН

Посвящается Проф. Мари и Акад. Давиду Тавхелидзе к их юбилею

1. Параллельно с систематической разработкой современного изложения начал математики [1] значительное количество исследований в последнее время посвящается новым отделам математики сложных структур, особенно сложным структурам теории и приложений математических уравнений. Сюда относятся, например, работы М. А. Лаврентьева и А. В. Бицадзе [2], М. Пиконе (M. Picone) [3], М. Кучма (M. Kuczma) [10], Р. Беллмана (R. Bellman) [4], авторов настоящей заметки [5] ... [8] и др.

Продолжая исследования математических систем сложных структур относящихся, в частности, к системам интегро-дифференциальных урзвнений с обыкновенными, полигармоническими (M. Picone), политепловыми (M. Nicolescu) и поливолновыми[1]) операторами, авторы рассматривают ниже решения систем интегро-дифференциальных уравнений с обыкновенными и тепловыми операторами.

2. Пусть задана система смешанных интегро-дифференциальных уравнений

$$u''(t) = f[t,\ u(t)] + \int_0^a K(t,\ \eta)\, w(t,\ \eta)\, \mathrm{d}\eta, \tag{1}$$

---

[1]) Поливолновые уравнения были названы многими учеными, как напр. Garrett Birkhoff, William Gordon и др., „уравнениями Манжерона" [9]

$$\frac{\partial w(t, x)}{\partial t} - \frac{\partial^2 w(t, x)}{\partial x^2} = F[t, x, u(t)] + \int_0^t P[t, x, \tau, u(\tau)]\, d\tau, \tag{2}$$

при дополнительных условиях

$$u(0) = u(h) = 0, \tag{3}$$

и

$$w(t, 0) = w(t, a) = 0, \quad w(0, x) = \varphi(x), \tag{4}$$

где f, K, F и P — заданные непрерывные функции в области $D\{0 \leqslant t \leqslant h,\ 0 \leqslant x \leqslant a,\ |u| \leqslant L_1,\ |w| \leqslant L_2\}$, L_1, L_2 — заданные числа, а функции f, F, P удовлетворяют в области D еще и условию Липшица по переменной $u(t)$. Пользуясь функцией $G(x, \xi, t-\alpha)$ [11], из (2), (4) находим

$$w(t, x) = \psi(t, x) + \int_0^t \int_0^a G(x, \xi, t-\alpha) \Big\{ F[\alpha, \xi, u(\alpha)] + \int_0^\alpha P[\alpha, \xi, \tau, u(\tau)]\, d\tau \Big\} d\xi\, d\alpha. \tag{5}$$

Пусть $H(t, \tau)$ — функция Грина краевой вадачи $u''(t) = 0$, $u(0) = u(h) = 0$. Тогда из (1), (3) получаем

$$u(t) = \int_0^h H(t, \tau) \Big\{ f[\tau, u(\tau)] + \int_0^a K(\tau, \eta)\, w(\tau, \eta)\, d\eta \Big\} d\tau. \tag{6}$$

Таким образом, задача (1) ... (4) свелась к системе нелинейных интегральных уравнений (5), (6). Из (6), с учётом (5), имеем

$$u(t) = \int_0^h H(t, \tau) \Big\{ f[\tau, u(\tau)] + \int_0^a K(\tau, \eta) \times \Big[\psi(\tau, \eta) + \int_0^\tau \int_0^a G(\eta, \xi, \tau-\alpha) \Big\{ F[\alpha, \xi, u(\alpha)] + \int_0^\alpha P[\alpha, \xi, \beta, u(\beta)]\, d\beta \Big\} d\xi\, d\alpha \Big] d\eta\, d\tau \Big\}. \tag{7}$$

Следовательно, если нелинейное интегральное уравнение (7) имеет единственное ремение $u(t) = \rho(t)$, то задача (1) ... (4) тоже имеет единственное решение

$$u(t) = \rho(t), \qquad w(t, x) = \psi(t, x) + \int_0^t \int_0^a G(x, \xi, t-\alpha) \times$$

(8)

$$\times \left\{ F[\alpha, \xi, \rho(\alpha)] + \int_0^\alpha P[\alpha, \xi, \tau, u(\tau)]\, d\tau \right\} d\xi\, d\alpha.$$

Очевидно, чтв если уравнения (7) реализуется принцип сжатых отображений, то задача (1) ... (4) имеет единственное решение в классе функций,обладающих непрерывными производными $u''(t)$, $\frac{\partial w(t, x)}{\partial t}$, $\frac{\partial^2 w(t, x)}{\partial x^2}$, и это решение можно построить итеративными методами.

3. Рассмотрим теперь случай, когда f, F и P — линейные операторы. В этом случае и. — д. у. (1), (2) запишутся в виде

$$u''(t) = f(t)\, u(t) + \int_0^a K(t, \eta)\, w(t, \eta)\, d\eta, \tag{9}$$

$$\frac{\partial w(t, x)}{\partial t} - \frac{\partial^2 w(t, x)}{\partial x^2} = F(t, x)\, u(t) + \int_0^t P(t, x, \tau)\, u(\tau)\, d\tau. \tag{10}$$

Пользуясь формулами (5) ... (7), получаем

$$w(t, x) = \psi(t, x) + \int_0^t \int_0^a S_1(x, \xi, t, \alpha)\, u(\alpha)\, d\xi\, d\alpha \equiv$$

(11)

$$\equiv \psi(t, x) + \int_0^t S(x, t, \alpha)\, u(\alpha)\, d\alpha,$$

$$u(t) = \sigma(t) + \int_0^h M(t, \tau)\, u(\tau)\, d\tau, \tag{12}$$

где ψ, S, M, σ — известные непрерывные функции. Из (12) следует, тя если 1 не является собственным значением ядра $M(t, \tau)$, то линейнао задача (9), (10), (3), (4) имеет единственное решение в рассматривае-

мом классе функций. Если 1 — собственное значение функции $M(t, \tau)$ ранга r, то либо задача (9), (10), (3), (4) не имеет искомого решения, либо это решение определяется неоднозначно и содержит r произвольных параметров.

4. Построим приближенные решения задачи (9), (10), (3), (4) для того случая, когда резольвента ядра $M(t, \tau)$ неизвестна. С этой целью воспользуемся выражением

$$\int_0^h M(t, \tau)\, u(\tau)\, d\tau = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} M(t, \tau)\, u(\tau)\, d\tau \approx \sum_{i=0}^{n-1} M(t, \tau_i) \int_{t_i}^{t_{i+1}} u(\tau)\, d\tau, \tag{13}$$

$$t_i \leqslant \tau_i \leqslant t_{i+1}.$$

Тогда из (12) получаем

$$\bar{u}(t) = \sigma(t) + \sum_{i=0}^{n-1} \gamma_i m_i(t), \tag{14}$$

где $\gamma_i = \int_{t_i}^{t_{i+1}} \bar{u}(\tau)\, d\tau$, $m_i(t) = M(t, \tau_i)$. Полагая в (14) $r_k = \int_{t_k}^{t_{k+1}} \sigma(t)\, dt$, и $m_{ik} = \int_{t_k}^{t_{k+1}} m_i(t)\, dt$, приходим к алгебраической системе уравнений

$$\gamma_k - \sum_{i=0}^{n-1} m_{ik} \gamma_i = r_k, \qquad (k = 0, 1, \ldots, n-1). \tag{15}$$

Если определитель системы (15) отличен от нуля, то эта система имеет единственное решение $\gamma_k = \lambda_k$, $(k = 0, 1, \ldots, n-1)$. Следовательно,

$$\bar{u}(t) = \sigma(t) + \sum_{i=0}^{n-1} \lambda_i m_i(t) \equiv g_n(t), \tag{16}$$

$$\bar{w}(t, x) = \psi(t, x) + \int_0^t S(x, t, \alpha)\, g_n(\alpha)\, d\alpha, \tag{17}$$

— искомое приближенное решение. Оценим величину допускаемой погрешности

$$u(t)-\overline{u}(t)=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[M(t,\ \tau)\,u(\tau)-M(t,\ \tau_i)\,\overline{u}(\tau)]\,\mathrm{d}\tau=$$

(18)

$$=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\{M(t,\ \tau)[u(\tau)-\overline{u}(\tau)]+[M(t,\ \tau)-M(t,\tau_i)]\,\overline{u}(\tau)\}\,\mathrm{d}\tau.$$

Если

$$l=1-\max_{[0,\ h]}\int_0^h|M(t,\ \tau)|\,\mathrm{d}\tau>0, \tag{19}$$

то из (18) находим

$$\|u(t)-\overline{u}(t)\|\leqslant\left\|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[M(t,\ \tau)-M(t,\ \tau_i)]\,\overline{u}(\tau)\,\mathrm{d}\tau\right\|:l=\rho(\Delta t_i). \tag{20}$$

Поэтому

$$|w(t,\ x)-\overline{w}(t,\ x)|\leqslant\left|\int_0^t|S(x,\ t,\ \alpha)|\,\mathrm{d}\alpha\right|\rho(\Delta t_i). \tag{21}$$

Из (20) и (21) видно, что если $\Delta t_i\to 0$, то приближенное решение задачи (9), (10), (3), (4) сходится к этому решению.

Замечания. 1°. Принимая во внимание формулу

$$\int_0^h M(t,\ \tau)|u(\tau)|\,\mathrm{d}\tau\approx\sum_{i=0}^{n-1}u(\tau_i)\int_{t_i}^{t_{i+1}}M(t,\ \tau)\,\mathrm{d}\tau\equiv\sum_{i=0}^{n-1}Q_i(t)\,u(\tau_i),$$

(22)

$$t_i\leqslant\tau_i\leqslant t_{i+1},$$

то, применив её к (12), получаем

$$\overline{u}(t)=\sigma(t)+\sum_{i=0}^{n-1}Q_i(t)\,\overline{u}(\tau_i). \tag{23}$$

Полагая в (23) $t=\tau_k\in[0,\ h]$, приходим к системе уравнений

$$\overline{u}(\tau_k)=\sigma(\tau_k)+\sum_{i=0}^{n-1}Q_i(\tau_k)\,\overline{u}(\tau_i),\qquad(k=0,\ 1,\ldots,n-1). \tag{24}$$

Если $\overline{u}(\tau_i) = d_i$, $(i = 0, 1, \ldots, n-1)$ — решение системы (24), то

$$\overline{u}(t) = \sigma(t) + \sum_{i=0}^{n-1} d_i Q_i(t), \tag{25}$$

$$\overline{w}(t, x) = \psi(t, x) + \int_0^t S(x, t, \alpha)\, \overline{u}(\alpha)\, \mathrm{d}\alpha \tag{26}$$

— приближенное решение рассматриваемой задачи.

2°. Погрешность оценивается аналогично предыдущему случаю:

$$\| u(t) - \overline{u}(t) \| \leqslant \left\| \sum_{i=0}^{n-1} \int_{k^i}^{k_{i+1}} M(t, \tau) - [\overline{u}(\tau) - \overline{u}(\tau_i)]\, \mathrm{d}\tau \right\| : l = \nabla_n, \tag{27}$$

$$| w(t, x) - \overline{w}(t, x) | \leqslant \left| \int_0^t | S(x, t, \alpha) | \mathrm{d}\alpha \right| \nabla_n, \tag{28}$$

при исловии, что неравенство (19) выполняется.

3°. Из оценок (27), (28) также следует, что построенный процесс (25), (26) сходится к точному решению задачу (9), (10), (3), (4) при $\Delta t_i \to 0$.

4°. Представляет интерес решение вопроса о возможном ослаблении требования (19).

5°. В одной из наших будущих заметок применим к вопросам о приближенных решениях рассматриваемых нами математических систем сложных структур ряд вышедших в последние годы работ по численным методам приближения высшего порядка точности нелинейных граничных задач, опубликованных в журнале „Numerische Mathematik“ P. G. Ciarlet, M. H. Schultz, R. S. Varga. См. также заметку авторов в печати в Rendiconti dell'Accademia Nazionale dei Lincei.

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SISTEME MATEMATICE CU STRUCTURĂ COMPLEXĂ

I. Asupra rezolvării sistemelor de ecuaţii integro-diferenţiale cu operatori ordinari şi calorici

(Rezumat)

Continuînd cercetările asupra sistemelor matematice cu structură complexă care includ în particular sistemele de ecuaţii integro-diferenţiale cu operatori ordinari poliarmonici, policalorici şi polivibranţi, autorii analizează soluţia unor sisteme de ecuaţii integro-diferenţiale cu operatori ordinari şi calorici.

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D. C. 519.4

ON M–PARTIALLY ORDERED GROUPS

BY

GH. COSTOVICI

1. A partially ordered group, G, which is at the same time an operator group with the operator domain M, so that, $a \leqslant b \Rightarrow ma \leqslant mb$, $m \in M$, is said to be an M-p. o. group. A subgroup H, of an M-p. o. group, G, is M-p. o. subgroup $\Leftrightarrow H$ is considered p. o. by trace and $mh \in H$, $m \in M$, $h \in H$.

H is M-p. o. convex subgroup $\Leftrightarrow H$ is M-p. o. subgroup and $h_1 \leqslant x \leqslant h_2$, $h_1, h_2 \in H \Rightarrow mx \in H$, $m \in M$. H is M-p. o. normal convex subgroup $\Leftrightarrow H$ is M-p. o. convex subgroup and $ghg^{-1} \in H$, $h \in H$, $g \in G$.

Since, the intersection of M-p. o. convex subgroups (M-p. o. normal convex subgroups) is M-p. o. convex subgroup (M-p. o. normal convex subgroup), exists the M-p. o. convex subgroup (M-p. o. normal convex subgroup), generated by $A \subseteq G$.

If we note with $M(A) = \{m_1 m_2 \dots m_r a\,;\ a \in A,\ m_i \in M,\ i = \overline{1, r}\}$, with $G(M(A)) = \{g.m_1 m_2 \dots m_r a.g^{-1};\ a \in A,\ m_i \in M,\ i = \overline{1, r},\ g \in G\}$ and say that x lies between y and $z \Leftrightarrow y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$, than, the M-p. o. convex subgroup (the M-p. o. normal convex subgroup) generated by $A \subseteq G$ is the set of the elements which lie between products of elements from

$$M(A) \cup M(A^{-1}) \cup A \cup A^{-1}(G(M(A)) \cup G(M(A^{-1})) \cup G(A) \cup G(A^{-1}).$$

Hence, the set of the M-p. o. convex subgroups (M-p. o. normal convex subgroups) forms a complete lattice, $S_1 = S_2(S_3)$, with respect to the set inclusion. We note that $\bigcup_{i \in I} H_i$, H_i being M-p. o. convex subgroup (M-p. o. normal convex subgroup), $i \in I$, is the set of the elements which lie between products of elements from $\bigcup_{i \in I} H_i$. Is easy to verify the above.

2. An equivalence relation, ρ, defined on a M-p. o. group, G, is said to be M-right compatible convex equivalence $\Leftrightarrow$

a) $x\rho y \Rightarrow xz\rho yz,\ z \in G$,

b) $x\rho y \Rightarrow mx\rho my,\ m \in M$, and

c) $e\rho h_1,\ e\rho h_2,\ h_1 \leqslant x \leqslant h_2,\ e$ being the unity of $G \Rightarrow e\rho x$.

The notion of M-left compatible convex equivalence (M-convex congruence) is obtained from the above definition if we replace a) with a'(a"):

a') $x\rho y \Rightarrow zx\rho zy,\ z \in G$,

a") $x\rho y \Rightarrow g_1xg_2\rho g_1yg_2,\ g_1$ and $g_2 \in G$.

Since, with respect to the order, $\rho_1 \leqslant \rho_2 \Leftrightarrow (x\rho_1 y \Rightarrow x\rho_2 y)$, from the set of the binary relations, the intersection of M-right compatible convex equivalences (M-left compatible convex equivalences) (M-convex congruences) is a M-right compatible convex equivalence (M-left compatible convex equivalence) (M-convex congruence), the set of M-right compatible convex equivalences (M-left compatible convex equivalences) (M-convex congruences) forms a complete lattice, $R_1(R_2)\,(R_3)$.

Considering a set $B_1(B_2)\,(B_3)$ of M-right compatible convex equivalences (M-left compatible convex equivalences) (M-convex congruences) and denoting by $G(B_i) = \{g\,; \exists\rho \in B_i,\ e\rho g\}$, $i = \overline{1, 3}$, $M(G(B_i)) = \{m_1m_2 \ldots m_rg;\ m_k \in M,\ k = \overline{1, r},\ \exists\rho \in B_i,\ e\rho m_1 m_2 \ldots m_rg\}$, $i = 1, 2$, $M(G(B_3)) = \{g.m_1 m_2 \ldots m_rg'.g^{-1};\ m_k \in M,\ k = \overline{1, r},\ \exists\rho \in B_3,\ e\rho g.m_1 m_2 \ldots m_rg'g^{-1}\}$, $C_i = G(B_i) \cup M(G(B_i))$, $i = \overline{1, 3}$, then, $a \bigvee_{\rho\in B_1} \rho\, b \Leftrightarrow b = xa$, x between products of elements from C_1, $a \bigvee_{\rho\in B_2} \rho b \Leftrightarrow b = ax$, x between products of elements from C_2, $a \bigvee_{\rho\in B_3} \rho b \Leftrightarrow b = xa$, x between products of elements from $C_3 \Leftrightarrow b = aa^{-1}xa = ay$, y between products of elements from C_3.

We shall make the proof only for $\bigvee_{\rho\in B_3} \rho$, the remaining of the demonstration following in the same way.

I. $a = ea,\ e \leqslant e \leqslant e,\ e\rho e,\ \rho \in B_3 \Rightarrow a \bigvee_{\rho\in B_3} \rho a,\ a \in G$.

II. $a \bigvee_{\rho\in B_3} \rho b \Leftrightarrow b = xa$, x between products of elements from $C_3 \Leftrightarrow a = x^{-1}b$, x^{-1} between products of elements from $C_3 \Leftrightarrow b \bigvee_{\rho\in B_3} \rho a$.

III. $a \bigvee_{\rho\in B_3} \rho b$ and $b \bigvee_{\rho\in B_3} \rho c \Leftrightarrow b = xa,\ c = yb$, x and y between products of elements from $C_3 \Leftrightarrow c = yxa$, yx between products of elements from $C_3 \Leftrightarrow a \bigvee_{\rho\in B_3} \rho c$.

IV. $a \bigvee_{\rho\in B_3} \rho b \Leftrightarrow b = xa$, x between products of elements from $C_3 \Leftrightarrow g_1bg_2 = g_1xag_2 = g_1xg_1^{-1}g_1ag_2 = yg_1ag_2$, y between products of elements from $C_3 \Leftrightarrow g_1\, ag_2 \bigvee_{\rho\in B_3} \rho g_1bg_2$.

V. $a \bigvee_{\rho \in B_3} \rho b \Leftrightarrow b = xa$, x between products of elements from $C_3 \Rightarrow$ $\Rightarrow mb = mx.ma$, $m \in M$, mx between products of elements from $C_3 \Leftrightarrow$ $\Leftrightarrow ma \bigvee_{\rho \in B_3} \rho mb$, $m \in M$.

VI. $e \bigvee_{\rho \in B_3} \rho h_1$, $e \bigvee_{\rho \in B_3} \rho h_2$, $h_1 \leqslant x \leqslant h_2 \Leftrightarrow h_1 = x_1 e = x_1$, x_1 between products of elements from C_3, $h_2 = x_2 e = x_2$, x_2 between products of elements from C_3 and $x_1 = h_1 \leqslant x \leqslant h_2 = x_2 \Rightarrow x = xe$, x between products of elements from $C_3 \Leftrightarrow e \bigvee_{\rho \in B_3} \rho x$.

VII. $a\rho b$, $\rho \in B_3 \Leftrightarrow e\rho ba^{-1}$, $\rho \in B_3 \Rightarrow b = ba^{-1} a = xa$, x between products of elements from $C_3 \Leftrightarrow a \bigvee_{\rho \in B_3} \rho b$. Therefore, $\rho \leqslant \bigvee_{\rho \in B_3} \rho$.

VIII. Let γ be a M-convex congruence so that $\rho \leqslant \gamma$, $\forall \rho \in B_3$, and let $a \bigvee_{\rho \in B_3} \rho b$ be. Then, $b = xa$ with $g_1.g_2 \dots g_s \leqslant x = ba^{-1} \leqslant g_1'.g_2' \dots g_t'$ and $e\rho_i g_i$, $\rho_i \in B_3$, $i = \overline{1, s}$, and with $e\rho_j' g_j'$, $\rho_j' \in B_3$, $j = \overline{1, t}$. $\rho \leqslant \gamma$, $\rho \in B_3 \Rightarrow$ $\Rightarrow e\gamma g_i$, $i = \overline{1, s}$, and $e\gamma g_j'$, $j = \overline{1, t} \Rightarrow e\gamma g_1.g_2 \dots g_s$ and $e\gamma g_1'.g_2' \dots g_t'$ which together with $g_1.g_2 \dots g_s \leqslant x = ba^{-1} \leqslant g_1'.g_2' \dots g_t' \Rightarrow e\gamma ba^{-1} \Leftrightarrow a\gamma b$.

3. The ordered triple (P, Q, f) denoting the mapping for which P is the domain, Q is the range and f is the correspondence law, we have the

Theorem. *The mapping* (R_i, S_i, f) *where* $f(\rho) = H_\rho = \{h;\ h \in G, e\rho h\}$ *and* $i = \overline{1, 3}$, *is a complete latticialy isomorphism.*

The parallelism permits make proof only for (R_3, S_3, f).

I. $e\rho e \Rightarrow e \in H_\rho$.

II. $h_1, h_2 \in H_\rho \Rightarrow e\rho h_1, e\rho h_2 \Leftrightarrow e\rho h_1, e\rho h_2^{-1} \Rightarrow e\rho h_1 h_2^{-1} \Leftrightarrow h_1 h_2^{-1} \in H_\rho$.

III. $h \in H_\rho \Leftrightarrow e\rho h \Leftrightarrow e\rho ghg^{-1} \Leftrightarrow ghg^{-1} \in H_\rho$.

IV. $h_1 \leqslant x \leqslant h_2$, $h_1, h_2 \in H_\rho \Leftrightarrow h_1 \leqslant x \leqslant h_2$, $e\rho h_1$, $e\rho h_2 \Rightarrow e\rho x \Leftrightarrow x \in H_\rho$.

V. $h \in H_\rho \Leftrightarrow e\rho h \Rightarrow e\rho mh$, $m \in M \Leftrightarrow mh \in H$, $m \in M$.

VI. Let $f(\rho_1) = f(\rho_2)$ be $\Leftrightarrow \{h;\ e\rho_1 h\} = \{h;\ e\rho_2 h\}$. Then, $a\rho_1 b \Leftrightarrow$ $\Leftrightarrow e\rho_1 a^{-1} b \Leftrightarrow e\rho_2 a^{-1} b \Leftrightarrow a\rho_2 b \Rightarrow \rho_1 = \rho_2$.

VII. $\forall H \in S_3$, $\exists \rho_H \in R_3$ defined so as: $x\rho_H y \Leftrightarrow x^{-1}y \in H$, so that, $f(\rho_H) = H$. $x^{-1}x = e \in H \Rightarrow x\rho_H x$. $x\rho_H y \Leftrightarrow x^{-1}y \in H \Leftrightarrow y^{-1}x \in H \Leftrightarrow y\rho_H x$. $x\rho_H y$ and $y\rho_H z \Leftrightarrow x^{-1}y$ and $y^{-1}z \in H \Rightarrow x^{-1}z \in H \Leftrightarrow x\rho_H z$. $x\rho_H y \Leftrightarrow x^{-1}y \in$ $\in H \Leftrightarrow g_2^{-1}x^{-1}yg_2 \in H \Leftrightarrow g_2^{-1}x^{-1}g_1^{-1}g_1yg_2 \in H \Leftrightarrow (g_1 xg_2)^{-1}g_1 yg_2 \in H \Leftrightarrow g_1xg_2\rho_H$ $g_1yg_2.x\rho_H y \Leftrightarrow x^{-1}y \in H \Rightarrow (mx)^{-1}my \in H$, $m \in M \Leftrightarrow mx\rho_H my$, $m \in M$. $e\rho_H h_1$, $e\rho_H h_2$, $h_1 \leqslant x \leqslant h_2 \Leftrightarrow e^{-1}h_1 = h_1 \in H$, $e^{-1}h_2 = h_2 \in H$, $h_1 \leqslant x \leqslant h_2 \Rightarrow x \in H \Leftrightarrow$ $\Leftrightarrow e^{-1}x = x \in H \Leftrightarrow e\rho_H x$. $f(\rho_H) = \{h;\ e\rho_H h\} = H$.

VIII. $f(\bigwedge_{i \in I} \rho_i) = H_{\bigwedge_{i \in I} \rho_i} = \{h;\ e \bigwedge_{i \in I} \rho_i h\} = \{h;\ e\rho_i h,\ \forall i \in I\} = \{h;\ h \in$ $\in H_{\rho_i},\ \forall i \in I\} = \bigcap_{i \in I} H_{\rho_i} = \bigcap_{i \in I} f(\rho_i)$.

IX. $f(\bigvee_{i\in I} \rho_i) = H_{\bigvee_{i\in I}\rho_i} = \{h\,;\, e \bigvee_{i\in I} \rho_i h\} = \{h\,;\, h = xe = x,\ x$ between products of elements from $C_3\} = \{h\,;\, h$ between products of elements from $\bigcup_{i\in I} H_{\rho_i}\} = \bigvee_{i\in I} H_{\rho_i} = \bigvee_{i\in I} f(\rho_i)$.

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DESPRE *M*-GRUPURI PARŢIAL ORDONATE

(Rezumat)

Se arată, ca rezultat principal, că mulţimea *M*-subgrupurilor parţial ordonate prin urmă, normal convexe, ale unui *M*-grup parţial ordonat, formează o latice completă prin incluziune, complet izomorfă cu laticea completă a *M*-congruenţelor convexe definite pe acelaşi *M*-grup parţial ordonat.

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C. Z. 512

ASUPRA UNEI CLASE DE ALGEBRE REALE DE ORDIN PAR

DE

I. ENESCU

1. Se ştie, [1], că o algebră asociativă şi comutativă de ordin finit cu scalari reali, care nu conţine idempotenţi diferiţi de unitatea principală şi de elementul nul, deci indecompozabilă, este suma (nedirectă) dintre un corp şi radicalul N al ei. Evident, în cazul scalarilor reali, acest corp nu poate fi decît cel real, R, sau corpul complex K, considerat desigur ca algebre reale.

Dacă existenţa algebrelor de forma $R + N$ este asigurată pentru orice ordin şi nu prezintă nici o dificultate în construcţia ei, nu acelaşi lucru se poate spune despre cazul unei algebre A care, avînd proprietăţile de mai sus, se prezintă sub forma unei sume dintre corpul complex K şi radicalul N. În [2] s-au studiat asemenea algebre, numai pentru cazul particular cînd N este algebra nulă.

Cercetarea cazului general al unei algebre reale A_n, de ordin n care se prezintă sub forma

$$A_n = K + N, \tag{1}$$

unde radicalul N este diferit de algebra nulă, constituie obiectul acestei Note.

2. Observînd că suma dintre K şi N, considerate ca spaţii liniare este directă, rezultă

$$\text{ordin } A_n = \text{ordin } K + \text{ordin } N. \tag{2}$$

Relaţia (2) permite alegerea în A_n a unei baze de forma

$$1,\ i,\ u_1,\ u_2,\ \ldots,\ u_{n-2}, \tag{3}$$

unde $1, i$ formează o bază pentru K iar $u_1, u_2, \ldots, u_{n-2}$ constituie o bază pentru radicalul N.

În baza (3), algebra A_n are o tabelă de înmulţire de forma

$$
(4)\qquad
\begin{array}{c|cccccc}
 & 1 & i & u_1 & u_2 & \cdots & u_{n-2} \\
\hline
1 & 1 & i & u_1 & u_2 & \cdots & u_{n-2} \\
i & i & -1 & \alpha_1^q u_q & \alpha_2^q u_q & \cdots & \alpha_{n-2}^q u_q \\
u_1 & u_1 & \alpha_1^q u_q & \cdot & \cdot & \cdot & \cdot \\
u_2 & u_2 & \alpha_2^q u_q & \cdot & & & \\
\vdots & \vdots & & \vdots & & & \\
u_{n-2} & u_{n-2} & \alpha_{n-2}^q u_q & \cdot & & &
\end{array}
$$

Elementele u_q, $(q = 1, 2, \ldots, n-2)$, se multiplică după o lege asociativă, satisfăcînd însă condiţiile care asigură faptul că radicalul N este o algebră de ordin $n-2$, nilpotentă şi diferită, conform ipotezei, de algebra nulă

$$
(5)\qquad u_p u_q = \gamma_{pq}^h u_h, \quad (p, q, h = 1, 2, \ldots, n-2).
$$

Se pune, în primul rînd, problema asociativităţii produsului pentru toate elementele din A_n. Observăm, în această ordine de idei că asociativitatea pentru elementele $1, i, 1, u_q$ şi u_q, u_p, $(p, q = 1, 2, \ldots, n-2)$, este asigurată.

Trebuie determinate constantele de structură

$$
(6)\qquad \alpha_\mu^\nu, \quad (\mu, \nu = 1, 2, \ldots, n-2),
$$

şi

$$
(7)\qquad \gamma_{pq}^h, \quad (p, q, h = 1, 2, \ldots, n-2),
$$

aşa fel încît elementele i, u_p pe de o parte şi i, u_p, u_q pe de altă parte, să aibă produsul asociativ.

Condiţiile de asociativitate de mai sus conduc la sistemul format din ecuaţiile

$$
(*)\qquad \begin{cases} (8) & i(i\,u_p) = -u_p, \\ (9) & i(u_p u_q) = (i\,u_p)u_q, \end{cases} \qquad (p, q = 1, 2, \ldots n-2).
$$

Observăm că în ecuaţiile (8) intervin numai constantele α_μ^ν care, o dată determinate şi înlocuite în (9), fac ca acestea din urmă să conţină numai necunoscutele γ_{pq}^h.

Studiul naturii sistemului (*) este dificil, dar pentru problema care ne interesează este suficientă constatarea existenţei a cel puţin unei soluţii reale sau, dimpotrivă, a absenţei oricărei asemenea soluţii.

În acest scop, observăm că ecuaţiile (8) formează, aşa cum s-a arătat în [2], un sistem echivalent cu ecuaţia matricială

(10) $$X^2 = -E,$$

unde

(11) $$X = \| \alpha_\mu^\nu \|, \quad (\mu, \nu = 1, 2, \ldots, n-2),$$

iar E este matricea unitate de tip $(n-2, n-2)$.

Utilizînd un rezultat stabilit în [2], urmează că, în cazul impar ($n = 2k+1$), ecuaţia (10) nu are nici o soluţie reală, fapt care demonstrează

Teorema 1. *În cazul impar, ($n = 2k+1$), nu există nici o algebră A_n cu structura* (1).

3. În cazul $n = 2k$, răspunsul la problema pusă este afirmativ.

Într-adevăr, algebra polinomială H_{2n}, generată de polinomul $(\lambda^2 + 1)^n$, $n \geqslant 3$, este indecompozabilă şi are structura unei sume de forma (1), unde N nu este algebra nulă, [3], [2] şi atunci are loc

Teorema 2. *Sistemul* (*) *are, în cazul $n = 2k$, ($k \geqslant 3$), cel puţin o soluţie reală.*

Cum algebrele H_{2n} răspund la toate condiţiile cerute [3], avem imediat

Teorema 3. *În cazul $n = 2k$, ($k \geqslant 3$), există algebre A_n cu structura (1) şi un asemenea exemplu este furnizat de algebra H_{2n}.*

4. Algebrele polinomiale H_{2n} nu sînt singurele care au proprietatea (1). Într-adevăr, să considerăm algebra D a numerelor duale în baza e_1, e_2 pentru care tabela de înmulţire este

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

şi o algebră, s-o notăm U_{2n}, indecompozabilă, comutativă şi care este suma dintre corpul complex K şi algebra nulă pe care s-o notăm prin $\mathfrak{N}$. Algebre de acest tip există şi ordinul lor este întotdeauna par, aşa cum s-a arătat în [2].

Deci considerăm algebra de ordin $2n$ cu structura

(12) $$U_{2n} = K + \mathfrak{N}.$$

Să formăm acum produsul direct $D \times U_{2n} = A_{4n}$. Dacă alegem în U_{2n} baza

(13) $$1, i, u_1, u_2, \ldots, u_{2n-2},$$

unde $1, i$ formează o bază pentru K, iar $u_1, u_2, \ldots, u_{2n-2}$ o bază pentru algebra nulă $\mathfrak{N}$ de ordin $2n-2$, atunci, o bază pentru produsul direct $A_{4n} = D \times U_{2n}$, este dată de

(14) $$\begin{cases} v_1 = 1.e_1, \quad v_2 = i.e_1, \quad v_3 = 1.e_2, \quad v_4 = i.e_2, \quad v_5 = e_1.u_1, \\ v_6 = e_1.u_2, \ldots, v_{2n+2} = e_1.u_{2n-2}, \quad v_{2n+3} = e_2.u_1, \\ v_{2n+4} = e_2.u_2, \ldots, v_{4n} = e_2.u_{2n-2}. \end{cases}$$

Relativ la baza (14) a algebrei A_{4n}, observăm următoarele :

1) $v_p v_q = 0$, $(p, q = 5, 6, \ldots, 2n)$;

2) $v_j v_h = 0$, $(j = 3, 4 ; h = 2n + 3, \ldots, 4n)$;

3) $v_3 v_p = v_{2n+s}$, $(p = 5, 6, \ldots, 2n + 2 ; s = 3, 4, \ldots, 2n)$;

4) produsele $v_4 v_q$, $(q = 5, 6, \ldots, 2n)$, sînt combinaţii liniare ale elementelor v_{2n+s}, $(s = 3, 4, \ldots, 2n)$.

Aceste observaţii stabilesc

L e m a 1. *Elementele* $v_3, v_4, \ldots, v_{2n+2}, v_{2n+3}, \ldots, v_{4n}$ *generează o algebră, fie ea* N, *de ordin* $4n - 2$, *care este subalgebră a algebrei* A_{4n}.

L e m a 2. *Subalgebra* N *este ideal al algebrei* A_{4n}.

L e m a 3. *Idealul* N *este maximal nilpotent pentru* A_{4n}, *deci este radicalul acesteia.*

În sfîrşit, observăm că algebra A_{4n} conţine cîmpul complex, pentru care elementele v_1 şi v_2 formează o bază.

Ne propunem acum să determinăm elementele idempotente ale algebrei A_{4n}.

Fie un element $x \in A_{4n}$, pe care să-l exprimăm în baza (14) :

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \sum_{i=3}^{4n} \beta_i v_i . \tag{15}$$

Apoi avem :

$$x^2 = (\alpha_1^2 - \alpha_2^2) v_1 + 2\alpha_1 \alpha_2 v_2 + \sum_3^{4n} \beta_i^2 v_i^2 + 2 \sum_{i,j=3}^{4n} \beta_i \beta_j v_i v_j +$$

$$+ 2\alpha_1 \sum_3^{4n} \beta_i v_i + 2\alpha_2 \sum_3^{4n} \beta_i v_2 v_i .$$

Condiţia $x^2 = x$ conduce la un sistem în necunoscutele $\alpha_1, \alpha_2, \beta_3, \ldots, \beta_{4n}$ ale cărui prime două ecuaţii sînt

$$\alpha_1^2 - \alpha_2^2 = \alpha_1, \quad 2\alpha_1 \alpha_2 = \alpha_2$$

şi care admit două soluţii reale, date de respectiv $\alpha_1 = 1$, $\alpha_2 = 0$ şi $\alpha_1 = \alpha_2 = 0$.

Dacă $\alpha_1 = \alpha_2 = 0$, urmează că $x \in N$ şi, cum radicalul este nilpotent, rezultă, a fortiori, că singurul său idempotent este elementul nul, deci $x = 0$. În particular, observăm că, în acest caz $\beta_3 = \beta_4 = \ldots = \beta_{4n} = 0$. Cu alte cuvinte, ecuaţiile sistemului $x^2 = x$, al căror membru stîng provine din $\sum_3^{4n} \beta_i v_i^2$ şi $2 \sum_{i,j=3}^{4n} \beta_i \beta_j v_i v_j$ au numai soluţia $\beta_3 = \beta_4 = \ldots = \beta_{4n} = 0$.

Dar, aceste ultime ecuaţii sînt aceleaşi şi în cazul $\alpha_1 = 1$, $\alpha_2 = 0$, ceea ce arată că, în această situaţie, $x = v_1$.

Am dovedit astfel

L e m a 4. *Algebra A_{4n} nu conţine idempotenţi diferiţi de unitatea principală şi de elementul nul, fiind deci indecompozabilă.*

Lemele 1 ,..., 4 demonstrează

T e o r e m a 4. *Algebra A_{4n} are structura* (1).

Se vede imediat că ordinul radicalului N est $4n - 2$, în timp ce ordinul pătratului său, N^2, conform observaţiilor precedente, este egal cu $2n - 2$. Notînd ordin N — ordin $N^2 = \delta$, avem că $\delta = 2n$.

Singurul caz pentru care $\delta = 2$, corespunde lui $n = 1$, adică produsului direct dintre algebra numerelor duale şi algebra numerelor complexe, situaţie exclusă de la sine, întrucît algebra U_{2n} nu mai are radical propriu. Pe de altă parte, $\delta = 2$ este singurul caz în care o algebră A_{4n} este polinomială, [3]. Am dovedit astfel

T e o r e m a 5. *Clasa algebrelor A_{4n} nu conţine nici o algebră polinomială.*

Teorema precedentă atrage imediat următorul

C o r o l a r. *Există cel puţin două tipuri neizomorfe de algebre cu structura* (1).

5. În încheiere, observăm următoarele :

a) Există pentru orice ordin finit, par, mai mare decît patru, algebre de tipul studiat şi acestea sînt polinomialele generate de $(\lambda^2 + 1)^n$.

b) Există algebre cu structura (1), de ordin $4n$, unde $n \geqslant 2$, care nu sînt polinomiale.

c) Rămîne deschisă problema determinării, pînă la izomorfism, a tipurilor de algebre cu structura (1).

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SUR UNE CLASSE D'ALGÈBRES RÉELLES D'ORDRE PAIR

(Résumé)

On démontre qu'une algèbre ayant la structure (1) où K est le corps complexe et N son radical qui n'est pas l'algèbre nulle, existe seulement si l'ordre n est pair. On construit des exemples non isomorphes de telles algèbres.

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EXISTENCE THEOREMS FOR GENERALIZED FREE PRODUCTS OF GROUPS (II) [1]

BY

ABDUL MAJEED

1. Introduction

In the first part of this paper, published in the same *Bulletin*, we have exposed, after a short introduction, notations and known facts, preliminaries concerned with the establishment of the existence theorems for generalized free products of groups. We give now some of such theorems.

2. Existence Theorems

2.1. Theorem 1. *Let A be the direct product of K and L and $B = \mathrm{gp}\{K, M\}$, $C = \mathrm{gp}\{L, M\}$ with M normal in both B and C. Let K' and L' be the groups generated by the automorphisms induced by K and L in M respectively. The generalized free product of A, B, C exists if and only if K' and L' commute elementwise.*

Proof. Since M is normal in both B and C and $K \cap M = L \cap M = 1$, we can regard B and C as split extensions of M by K and of M by L respectively, so that there exist homomorphisms $\varphi: K \to K'$ and $\psi: L \to L'$. Now let the generalized free product F of A, B, C exist, then, as M is normalized by K and L, and F is generated by K, L and M, M is normal in F. Moreover the factor group F/M is isomorphic to the group generated by K and L i. e. to A. Thus we can regard F as an extension of M by A. Let A' be the group generated by the automorphisms induced by A in M, then $A' = \mathrm{gp}\{K', L'\}$ and there is a homomorphism $\varphi^*: A \to A'$ which coincides with φ on K and with ψ on L. Since in A, $[k, l] = 1$, therefore,

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[1]) This paper forms a part of a thesis submitted to the Australian National University for the degree of Master of Arts, 1966.

$$1 = [k, l]\varphi^* = [k\varphi^*, l\varphi^*] = [k', l'] \quad \text{for all} \quad k' \in K', l' \in L',$$

that is K' and L' commute elementwise.

Conversely suppose that the automorphisms induced by K and L in M commute. As every embedding of the amalgam of A, B, C automatically embeds the amalgam of B and C, the desired group if it exists, must be isomorphic to a factor group of F^A, the generalized free product of B and C, using the maximality of the free embedding. In this factor group F^A/N^*, (say) K and L become elementwise permutable. Now in F^A, K and L generate their free product $\overline{F}^A$ (cf. [3, Theorem 4.1]). If N is the normal closure of the set of all commutators $[k, l]$, $k \in K$, $l \in L$ in $\overline{F}^A$, then $\overline{F}^A/N \cong A$. Therefore N^* must contain N and if N^* is the least normal subgroup of $\overline{F}^A$ containing N, F^A/N^* will be the free embedding of the given amalgam provided that it is an embedding at all. Thus the free product of an amalgam exists if and only if

i) $N^* \cap \overline{F}^A = N$ so that $\overline{F}^A N^*/N^* \cong \dfrac{\overline{F}^A}{\overline{F}^A \cap N^*} \cong \dfrac{\overline{F}^A}{N} \cong A$;

ii) N^* is tidy with respect to K and L in F^A;

iii) N^* contains no element of the form fb or fc, $b \neq 1 \in B$, $c \neq 1 \in C$, $f \neq 1 \in \overline{F}^A$, so that in F^A/N^*, the image A of $\overline{F}^A$ has the correct intersection with B with C.

Let m be an element of M. As N is generated by $[k, l]$, $k \in K$, $l \in L$, and normal in $\overline{F}^A$ (by Golovin [5]) the elements $m^{-1}(k, l)m$ generate N^*. We show that these are elements of N.

Let α_k, β_l be the automorphisms induced by k^{-1} and l^{-1} on M, that is for all $m \in M$, $m\alpha_k = kmk^{-1}$, and $m\beta_l = lml^{-1}$, then,

$$m^{-1}[k, l]\, m = m^{-1}\, k^{-1}\, l^{-1}\, klm = m^{-1}\, k^{-1}\, l^{-1}\, k\,(m\,\beta_l)\, l = m^{-1}\, k^{-1}\, l^{-1}(m\,\beta_l\,\alpha_k)kl =$$
$$= m^{-1}(m\beta_l\,\alpha_k\,\beta_{l-1}\,\alpha_{k-1})\, k^{-1}\, l^{-1}\, kl.$$

But $\beta_{l-1} = \beta_l^{-1}$, $\alpha_{k-1} = \alpha_k^{-1}$ and $[\alpha_k, \beta_l] = 1$, therefore $m^{-1}[k, l]m = [k, l] \in N$, so that $N^* = N$ proving (i).

To prove (ii) we note that each element of B is uniquely of the form km, $k \in K$, $m \in M$, and similarly every element of C is uniquely of the form lm, $l \in L$, $m \in M$. Now assume that N^* contains an element of the form $bc = km_1lm_2 = klm_1'm_2$.

As $N^* \subseteq \overline{F}^A$, $bc \in \overline{F}^A$, $kl \in \overline{F}^A$, hence $m_1'\, m_2 \in \overline{F}^A$. But as K and L both intersect M trivially, it follows ([3, theorem 4.1]) that $\overline{F}^A$ also intersects M trivially. Hence $m_1'\, m_2 = 1$ and so $bc = kl \in N^* = N$. But N is tidy in $\overline{F}^A$ being the kernel of the mapping from $\overline{F}^A$ to $K \times L$. Hence $k = l = 1$ that is $bc = 1$. This proves (ii).

To see that (iii) holds, assume $n = b\bar{f}$ for some $n \in N^*$, $b \in B$, $\bar{f} \in \bar{F}^A$, then as n and $\bar{f}$ are both in $\bar{F}^A$, b also lies in $\bar{F}^A$ that is $b \in K$. Similarly $n = c\bar{f}$, $c \in C$, $f \in \bar{F}^A$ implies $c \in L$, so that no element of N^* can be written in the above form. This completes the proof of the theorem.

Remark. Here again we remark that A can be taken as the generalized direct product of K and L amalgamating a central subgroup Z, and further that if A, B and C are finite, their generalized free product is also finite.

However, there exist embeddable amalgams of three groups of the form described in theorem 1 but with a different (seemingly weaker) restriction on A, namely we suppose that instead of A being the direct product of K and L, there is a homomorphism of A onto the group generated by the automorphisms induced by K and L in M. It is interesting to note that this condition again turns out to be both necessary and sufficient for the existence of the generalized free products of such groups. Thus we prove :

Theorem 2. *Let $A = \mathrm{gp}\{K, L\}$, $B = \mathrm{gp}\{K, M\}$, $C = \mathrm{gp}\{L, M\}$ with M normal in both B and C. The generalized free product of A, B, C exists if, and only if, there is a homomorphism φ of A onto the group generated by the automorphisms induced by K and L in M.*

Proof. Let K', L' be the groups generated by the automorphisms induced by K and L in M respectively. Then there exists homomorphisms $\varphi_1 : K \to K'$ and $\varphi_2 : L \to L'$. Let φ be the given homomorphism of A onto $A^* = \mathrm{gp}\,\{K', L'\}$ which coincides with φ_1 on K and with φ_2 on L. We form the extension G of M by A determined by the homomorphism φ ; that is the group of pairs (a, m), $a \in A$, $m \in M$ where

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, m_1^{a_2\varphi} m_2),$$

a_1, $a_2 \in A$, $m_1, m_2 \in M$ and show that this group embeds the amalgam.

Since $\varphi/K = \varphi_1$, $\varphi/L = \varphi_2$, in G the groups B_1, C_1, consisting of the pairs (k, m), $k \in K$, $m \in M$ and of (l, m), $l \in L$, $m \in M$ are isomorphic to B and C and intersect precisely in the group of pairs $(1, m)$, $m \in M$, isomorphic to M. That these groups have the right intersection also with the group A, consisting of the pairs $(a, 1)$, $a \in A$ isomorphic to A, is obvious. Thus G embeds the amalgam of A, B, C.

Conversely, suppose the generalized free product F of A, B, and C exists. Since M is normalized by K and L, and F is generated by K, L and M, therefore M is normal in F and moreover $F/M \cong \mathrm{gp}\{K, L\} = A$ so that F is an extension of M by A. Hence there is a homomorphism of A onto the group generated by the automorphisms induced by K and L in M. The proof of the theorem is now complete.

Remark. It is worth mentioning that here again, the generalized free product of groups obeying conditions as those in the above theorem is finite if the groups themselves are finite.

2.2. We now turn to investigate the embeddability of an amalgam of three groups of a different kind. This amalgam consists of three dihedral groups of order $2l$, $2m$, and $2n$ respectively and taken in the special form as:

$$(1)\qquad \begin{cases} A = \mathrm{gp}\{a, b\,;\ a^2 = b^2 = (ab)^l = 1\}, \\ B = \mathrm{gp}\{b, c\,;\ b^2 = c^2 = (bc)^m = 1\}, \\ C = \mathrm{gp}\{c, a\,;\ c^2 = a^2 = (ca)^n = 1\}. \end{cases}$$

The amalgam **A** formed by these groups is their reduced amalgam. If any two of the l, m and n are equal to 2 then this amalgam falls under the category of those considered in theorem 2. The following theorem shows that the above amalgam is embeddable for all values of l, m and n.

Theorem 3. *The generalized free product of an amalgam* **A** *of groups of type* A, B, C *in* (1) *exists.*

Proof. In view of the above remark, we need discuss the existence of the generalized free product for m and n not both equal to 2. Following the scheme of proof of theorem 1, we first take the generalized free product $\bar{F}^A$ of B and C and knowing that the subgroup $\bar{F}^A$ of F^A, generated by K and L, and in this case by the elements a and b, is their free product, we take the normal closure N of the group generated by the left hand sides of the relations of A and their conjugates. If N^* is the normal closure of N in F^A and satisfies the conditions (i), (ii) and (iii) in the proof of theorem 1, the factor group F^A/N^* will then be isomorphic to the generalized free product of A, B, C. Thus the problem of existence of the generalized free product of **A** reduces to that of showing that N^* possesses the properties (i), (ii) and (iii) mentioned above.

The generalized free product of B and C is

$$F^A = \mathrm{gp}\{a, b, c\,;\ a^2 = b^2 = c^2 = (bc)^m = (ca)^n = 1\}.$$

Putting $bc = g$, $ca = h$, F^A is seen to have an alternative representation

$$F^A = \mathrm{gp}\{g, h, c\,;\ g^m = h^n = (gh)^2 = c^2 = (gc)^2 = (ch)^2 = 1\}$$

as the free product of

$$B' = \mathrm{gp}\{g, c\,;\ g^m = c^2 = (gc)^2 = 1\} \cong B \text{ and } C' = \mathrm{gp}\{h, c\,;\ h^n = c^2 = (ch)^2 = 1\} \cong C$$

amalgamating c. In F^A, a and b generate their free product which is the infinite dihedral group and has, as its homomorphic images, all the finite dihedral groups. In particular, since $a^{-1}.ba.a = ab = (ba)^{-1}$, $b^{-1}.b.a.b = ab = (ba)^{-1}$ the group generated by $(ba)^l$, $l \geqslant 2$ is a normal subgroup of $\bar{F}^A$. Taking this as N we have $\bar{F}^A/N \cong A$. Let N^* be the normal closure of N in F^A. N^* is then generated by $(ba)^l$ and $c(ba)^l c$. We first show that N^* is a free group of rank 2.

Write $ba = gh$, $c(ba)c = c(gh)c = g^{-1}h^{-1} = (hg)^{-1}$, we see that N^* is generated by $(gh)^l$ and $(hg)^{-l}$. These two are independent generators of N^*, for if $(gh)^l = (hg)^{-pl}$ for some $p>1$, ($p \neq 1$ because $c(gh)c = c^{-1}(gh)c = (g^{-1}h^{-1}) \neq (gh)$), then

$$(gh)^l = (hg)^{-pl} = (g^{-1}h^{-1})^{pl} = g^{-1}(h^{-1}g^{-1})g^{pl} = g^{-1}(gh)^{-pl}g$$

that is $g(gh)^l g^{-1} = (gh)^{-pl}$ and hence $g^2(gh)g^2 = (gh)^{p^2 l}$. Continuing in this way, we have

$$g^m(gh)^l g^{-m} = (gh)^l = (gh)^{(-1)^m p^m}.$$

Thus $(gh)^{((-1)^m p^m - 1)l} = 1$. Since $p \neq 1$, this relation implies that gh is of finite order, a contradiction. Further, in the second representation of F^A, g and h generate their free product. A word w of the form

$$(gh)^{\varepsilon_1 l}(hg)^{\delta_1 l} \dots (gh)^{\varepsilon_k l}(hg)^{\delta_k l}$$

cannot reduce to identity because in w there are no cancellations but only amalgamations of the form $g.g$, $h.h$, $g^{-1}.g^{-1}$, $h^{-1}.h^{-1}$ and since not both m and n are equal to 2, these amalgamations cannot reduce the above word to identity. Thus there is no non-trivial relation of the form $w = 1$. The group N^* generated by $(gh)^l$ and $c(gh)^l c$ is, therefore, free of rank 2.

Now $c(gh)^l c = c(ba)^l c$ does not belong to $\bar{F}^A$, and moreover $N^* \cap \bar{F}^A$, as a subgroup of a free group is free, by Schreier's theorem; as a subgroup of the infinite dihedral group $\bar{F}^A$ it can be free only if it is an infinite cycle; as it contains N, this cycle must be generated by $(gh)^{l_1}$ where l_1/l. But the free group with free generator $(gh)^l$ contains no proper root of this. Hence $l_1 = l$ and $N^* \cap \bar{F}^A = N$ so that condition (i) is satisfied.

Also N^* is tidy with respect to B and C, for every element of N^* is of the form

$$n^* = (gh)^{\varepsilon_1 l}(hg)^{\delta_1 l} \dots (gh)^{\varepsilon_k l}(hg)^{\delta_k l}.$$

In n^*, c occurs an even number of times and since every element of B is of the form cg^k, ($k = 0, 1, \dots, m-1$), $n^* \neq cg^k$, because then n^* like cg^k will be of finite order. Similarly $N^* \cap C = 1$. This proves (ii).

Also no element of N^* can be written in the form $fb^*f \in \bar{F}^A$; $b^* \in B$, for f is of the form $a^\varepsilon bab \dots ab^\delta$ and $b^* = c(bc)^u$, ($u = 0, 1, \dots, m-1$), and in fb^*, between no two consecutive c's the element a of A occurs whereas in any n^* it does. Similarly $n^* \neq fc^*$, $c^* \in C$. This proves (iii) and completes also the proof of the theorem.

Remark. The discussion of this particular amalgam has been inserted at this point because the argument is close to the earlier arguments in this paragraph. It arose out of my endevour to show that the group

$$F = \mathrm{gh}\{a, b, c; a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ca)^n = 1\}$$

described by Coxeter and Moser (cf [7], pp. 37—39]) as the group of reflexions in the sides of a spherical triangle with angles π/l, π/m, π/n does in fact embed the amalgam of three dihedral groups and therefore is their free product. F is known to be finite if $1/l + 1/m + 1/n > 1$ and infinite otherwise. If we write F as

$$F = \mathrm{gh}\{g, h, c;\ g^m = h^n = (gh)^l = c^2 = (gc)^2 = (ch)^2 = 1\}$$

then it is easy to see that F is a split extension of

$$P = P(m, n, l) = \mathrm{gp}\{g, h;\ g^m = h^n = (gh)^l = 1\}$$

by a cyclic group of order 2. P belongs to the well known family of groups called the *polyhedral groups*. It possesses, among its factor groups, many of the known finite simple groups (cf. [6]).

2.3. Let **A** be the reduced amalgam of three groups H_1, H_2, and H_3 such that $H_i \cap H_j = H_{ij} = H_{ji}$ for all $i, j = 1, 2, 3, i \neq j$. These groups can be taken as

$$H_1 = \mathrm{gp}\{H_{12}, H_{13}\},\ H_2 = \mathrm{gp}\{H_{12}, H_{23}\},\ H_3 = \mathrm{gp}\{H_{13}, H_{23}\}.$$

It is known that the amalgam **A** is embeddable in a group if any one of the H_i' s, H_3 say, is the generalized free product of H_{j3}, $(j = 1, 2)$ [4, theorem 9.0]. Hanna Neumann has also shown that, in general, this theorem cannot be generalized to the case of more than three groups. The examples she has constructed deal with the most general type of group amalgams. However if we have an amalgam of more than three groups with a limited number of intersections, the theorem can be generalized.

The kind of amalgam we consider involves n groups $G_1, G_2, \ldots G_n$ where for every $i = 1, 2, \ldots n$ with $G_{n+1} = G_1$, we have,

$$G_i \cap G_{i+1} = H_i, \text{ and } G_i \cap G_j = H \text{ for } j \neq i \pm 1.$$

For $n = 3$ this is the same as the amalgam considered above. Any amalgam of the above form shall be called *of a special type* or more briefly *of type S*. While for $n = 3$ such an amalgam may not be embeddable without one or other additional restriction, we show that for $n > 3$, no further condition is required.

Theorem 4. *Let n given groups $G_1, G_2, \ldots G_n$; $n > 3$ form an amalgam* **A** *of type S. Then* **A** *is embeddable in a group, that is, the generalized free product of $G_1, G_2, \ldots, G_n$ exists.*

Proof. Since we need consider only the reduced amalgam, it can be supposed that the amalgam **A** formed by $G_1, G_2, \ldots G_n$ is already reduced. Then, under the assumption that $G_{n+1} = G_1$, $H_{n+1} = H_1$, each G_{i+1} is generated by its subgroups H_i, H_{i+1} $(i = 1, 2, \ldots, n)$. These groups are arranged in pairs as follows:

When n is even, $n = 2m$, as :

$$(G_1, G_{2m}), (G_2, G_{2m-1}), \ldots, (G_i, G_{2m-i+1}), \ldots, (G_m, G_{m+1}) \tag{2}$$

and when $n = 2m + 1$, as :

$$(G_1, G_{2m+1}), \ldots, (G_i, G_{2m-i+2}), \ldots, (G_{m-1}, G_{m+3}) \tag{3}$$

together with the triplet (G_m, G_{m+1}, G_{m+2}). Then the free product of each pair exists by Schreier's theorem: We postpone the consideration of the triplet (G_m, G_{m+1}, G_{m+2}).

In the case of (2), the free product F_i of (G_i, G_{2m-i+1}) amalgamating H for $i = 2, 3, \ldots, m-1$, and H_{2m} or H_m for $i = 1$ or m respectively, are such that in each of these :

a) two consecutive groups H_i, H_{i+1} generate the group G_{i+1};

b) any pair H_i, H_j ; $j \neq i \pm 1$ generate their free product amalgamating H.

To substantiate this claim, it is sufficient to prove it for F_1 only. Since F_1 is the generalized free product of

$$G_1 = \mathrm{gp}\{H_{2m}, H_1\}, \quad G_{2m} = \mathrm{gp}\{H_{2m-1}, H_{2m}\},$$

a) is obviously satisfied. Also by the properties of the free product as $H_1 \subseteq G_1$, $H_{2m-1} \subseteq G_{2m}$ and $H_1 \cap H_{2m-1} \subseteq G_1 \cap G_{2m} = H_{2m}$. We have, $H_1 \cap H_{2m-1} = H_1 \cap H_{2m} \cap H_{2m-1} = (H_1 \cap H_{2m}) \cap (H_{2m} \cap H_{2m-1}) = H \cap H = H$, hence by [4, theorem 5.0], H_1, H_{2m-1} generate their free product amalgamating H, so that F_1 has b) also.

Take now the first two of the free products F_i, $(i = 1, 2, \ldots, m)$

$$F_1 = \mathrm{gp}\{H_1, H_{2m-1}, H_{2m}\}, \qquad F_2 = \mathrm{gp}\{H_1, H_2, H_{2m-2}, H_{2m-1}\}.$$

The groups H_1 and H_{2m-1} are contained in F_1 and F_2 and generate in both their free product amalgamating H. The free product $F^{(1)}$ of F_1 and F_2 amalgamating $\{H_1 * H_{m-1} ; H\}$ can therefore be formed. In $F^{(1)}$, since $H_{2m} \subseteq G_{2m}$, $H_2 \subseteq G_2$, and $H_2 \cap H_{2m} \subseteq G_2 \cap G_{2m} = H$ we have

$$H_2 \cap H_{2m} = H_2 \cap H \cap H_{2m} = (H_2 \cap H) \cap (H \cap H_{2m}) = H.$$

Similarly, $H_{2m-2} \cap H_{2m} = H$. Thus for $F^{(1)}$ also the conditions a) and b) are satisfied. Form now the generalized free product $F^{(2)}$ of $F^{(1)}$ and F_3 amalgamating $\{H_2 * H_{2m+2} ; H\}$ and continuing in this way, lastly, the free product $F^{(m-1)}$ of $F^{(m-3)}$ and F_{m-1} amalgamating $\{H_{m-2} * H_{m+2} ; H\}$. As it has been shown that $F^{(1)}$ satisfies the requirements a) and b), we have a basis for induction. Suppose that we have already proved that $F^{(m-2)}$ has the properties a) and b). Then, since $\{H_{m-1} * H_{m+1} ; H\} = H^*$ (say) is contained in both $F^{(m-2)}$ and F_m, the free product $F^{(m-1)}$ of the amalgam $\mathbf{A}' = \mathrm{am}\,(F^{(m-1)}, F_m ; H^*)$ may be formed. Now for $i \neq m, m-1, m+1$,

$$H_i \cap H_m \subseteq F^{(m-2)} \cap F_m = H^*.$$

However, by assumption, $H_i \cap H_{m-1} = H_i \cap H_{m+1} = H$; and also in F_m, $H_m \cap H_{m-1} = H_m \cap H_{m+1} = H$, therefore $H_i \cap H^* = H^* \cap H_m = H$. Thus $H_i \cap H_m = H_i \cap H^* \cap H_m = (H_i \cap H^*) \cap (H^* \cap H_m) = H$. Hence $F^{(m-1)}$ satisfies conditions a) and b). But then $F^{(m-1)}$ contains all the G_i' s with their precise intersections and therefore embeds their amalgam.

For the odd case $n = 2m + 1$, we show that the generalized free product of the triplet (G_m, G_{m+1}, G_{m+2}) exists provided that $n > 3$, that is $m > 1$. For this, we have, in view of [4, theorem 5.0], only to note that the reduced amalgam of these groups consists of just the group G_{m+1} and hence is embeddable, so that F_m exists. Infact if $X = \{G_m * G_{m+1}; H_m\}$ then $F_m = \{X * G_{m+1}; H_{m+1}\}$ and in F_m the groups H_{m-1} and H_{m+2} generate their free product. That the free product $F^{(m-1)}$ of $F^{(m-2)}$ and F_m amalgamating $\{H_{m-1} * H_{m+2}; H\}$ and so also of $G_1, G_2, \ldots, G_n$ exists, follows in exactly the same way as above.

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TEOREME DE EXISTENŢĂ PENTRU PRODUSE LIBERE GENERALIZATE DE GRUPURI (II)

(Rezumat)

După ce în [1] s-au introdus notaţiile, au fost reamintite unele rezultate cunoscute şi s-au expus o serie de lucruri preliminare, în lucrarea de faţă se demonstrează un număr de teoreme de existenţă pentru produse libere generalizate de două şi trei grupuri, precum şi pentru produsul liber al unui amalgam de mai multe grupuri.

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KRYLOFF FUNCTIONS ON ALGEBRAS

BY

ALFRED BRAIER

1. Introduction

In the previous papers [1], [2], the author introduced exponential functions as well as trigonometric and hyperbolic functions on an algebra A_n on the real field R, by mposing, respectively, that for $\forall w \in A_n$,

$$\frac{d}{dx} f(xw) = wf(xw), \qquad \frac{d^2}{dx^2} T(xw) = \mp w^2 T(xw), \tag{1}$$

with $x \in R$.

If the algebra A_n is associative, commutative and has the unit w^0, then

$$\left\{\begin{aligned} \cos(xw) &= \frac{1}{2}[\exp(xpw) + \exp(-xpw)], \\ \sin(xw) &= \frac{1}{2p}[\exp(xpw) - \exp(-xpw)], \end{aligned}\right. \quad \text{with } p^2 = -w^0; \tag{2}$$

$$\left\{\begin{aligned} \cosh(xw) &= \frac{1}{2}[\exp(xqw) + \exp(-xqw)], \\ \sinh(xw) &= \frac{1}{2q}[\exp(xqw) - \exp(-xqw)], \end{aligned}\right. \quad \text{with } q^2 = w^0. \tag{3}$$

Also, if one utilises as basis in such an algebra, the powers of element v of which minimal polynomial coincides with the characteristic polynomial of the ordinary differential equation with constant coefficients

$$\varphi^{(n)} + a_1\varphi^{(n-1)} + \dots + a_{n-1}\varphi' + a_n\varphi = 0, \tag{4}$$

then the components of $\exp(xv)$ in the cyclical basis are fundamental solutions of (4).

In a similar manner we can introduce various functions on algebras, by means of differential equations with constant coefficients, like (1). Then, if the expressions of the fundamental solutions of such equations on the

real field are known, one can construct easily these solutions for the elements w of different algebras, by utilising the exponential, trigonometric and hyperbolic functions previously obtained for these elements. The problem reduces finally to the calculation of $\exp(xw)$, $\exp(xpw)$ $(p^2 + w^0 = 0)$ and $\exp(xqw)$ $(q^2 - w^0 = 0)$, i. e. implies the solving of some ordinary differential equations with constant coefficients, like (4).

We shall illustrate this idea by defining Kryloff functions on algebras, which among other results, provides also some direct relations between the usual Kryloff functions (on the real field).

2. Kryloff Functions on Algebras

Kryloff functions

$$(5)\quad \begin{cases} \varphi(x) = \cos x \cosh x, & \psi(x) = \dfrac{1}{2}(\sin x \cosh x + \cos x \sinh x), \\ \theta(x) = \dfrac{1}{2} \sin x \sinh x, & \chi(x) = \dfrac{1}{4}(\sin x \cosh x - \cos x \sinh x) \end{cases}$$

are fundamental solutions of equation $\Phi^{IV} + 4\Phi = 0$, i. e. components of the exponential function [3]

$$(6)\qquad \exp(xv_2) = \varphi(x)v_1 + \psi(x)v_2 + \theta(x)v_3 + \chi(x)v_4$$

build up for element v_2 of the cyclical basis

$$(7)\qquad v^0 = v_1, \quad v = v_2, \quad v^2 = v_3, \quad v^3 = v_4 \quad \text{and} \quad v^4 = -4v$$

of $\mathcal{P}$-polynions algebra (cartesian product of two complex algebras) (v_1 is the unit of that algebra).

For the other two elements of basis (7), the exponential functions are respectively

$$(8)\quad \begin{cases} \exp(xv_3) = \cos 2x\, v_1 + \dfrac{1}{2} \sin 2x\, v_3, \\ \exp(xv_4) = \varphi(2x)v_1 + 2\chi(2x)v_2 - \theta(2x)v_3 + \dfrac{1}{2}\psi(2x)v_4. \end{cases}$$

Also, using (2), with $p = \pm v_3/2$ and (3), with $q = \pm v_1$, we can obtain [2]

$$(9)\quad \begin{cases} \cos(xv_2) = \varphi(x)v_1 - \theta(x)v_3, \\ \sin(xv_2) = \psi(x)v_2 - \chi(x)v_4, \end{cases} \quad \begin{cases} \cosh(xv_2) = \varphi(x)v_1 + \theta(x)v_3, \\ \sinh(xv_2) = \psi(x)v_2 + \chi(x)v_4; \end{cases}$$

$$(10)\quad \begin{cases} \cos(xv_3) = \cosh 2x\, v_1, \\ \sin(xv_3) = \dfrac{1}{2}\sinh 2x\, v_3, \end{cases} \quad \begin{cases} \cosh(xv_3) = \cos 2x\, v_1, \\ \sinh(xv_3) = \dfrac{1}{2}\sin 2x\, v_3, \end{cases}$$

and

$$(11)\quad \begin{cases} \cos(xv_4) = \varphi(2x)\,v_1 + \theta(2x)\,v_3, \\ \sin(xv_4) = -\,2\chi(2x)\,v_2 + \dfrac{1}{2}\,\psi(2x)\,v_4, \end{cases} \qquad \begin{cases} \cosh(xv_4) = \varphi(2x)\,v_1 - \theta(2x)\,v_3, \\ \sinh(xv_4) = 2\chi(2x)\,v_2 + \dfrac{1}{2}\,\psi(2x)\,v_4. \end{cases}$$

Let now $K(xw)$ be the element of an commutative ans associative algebra A_n, with unit, on the real field R, which we put in correspondence with xw $(w \in A_n)$, for a fixed $x \in R$. If, for any w

$$(12)\qquad \frac{d^4}{dx^4}K(xw) = -\,w^4K(xw),$$

then $K(xw)$ is to be called a generalized Kryloff function on algebra A_n.

Let also be four functions, namely $\varphi(xw)$, $\psi(xw)$, $\theta(xw)$ and $\chi(xw)$, which verify (12), and such that

$$(13)\quad \begin{cases} \varphi(0) = w^0, & \psi(0) = 0, & \theta(0) = 0, & \chi(0) = 0, \\ \varphi'(0) = 0, & \psi'(0) = w, & \theta'(0) = 0, & \chi'(0) = 0, \\ \varphi''(0) = 0, & \psi''(0) = 0, & \theta''(0) = w^2, & \chi''(0) = 0, \\ \varphi'''(0) = 0\,; & \psi'''(0) = 0\,; & \theta'''(0) = 0\,; & \chi'''(0) = w^3, \end{cases}$$

where w^0 is the unit of A_n, and $(\ \ )' = \dfrac{d}{dx}(\ \ )$.

These functions will be called fundamental solutions of Eq. (12), or more simple, *Kryloff functions on* A_n.

If $w = w^0$ (unit of A_n), then $\varphi(xw^0)=\varphi(x)w^0$, $\psi(xw^0)=\psi(x)w^0$, $\theta(xw^0)= = \theta(x)w^0$, $\chi(xw^0) = \chi(x)w^0$, where $\varphi, \ldots, \chi$ are the usual Kryloff functions.

It is clear that

$$(14)\quad \begin{cases} \varphi(xw) = \cos(xw)\cosh(xw), \\ \psi(xw) = \dfrac{1}{2}\,[\sin(xw)\cosh(xw) + \cos(xw)\sinh(xw)], \\ \theta(xw) = \dfrac{1}{2}\sin(xw)\sinh(xw), \\ \chi(xw) = \dfrac{1}{4}\,[\sin(xw)\cosh(xw) - \cos(xw)\sinh(xw)] \end{cases}$$

satisfy (12) under conditions (13), that is they can be utilised in order to obtain the Kryloff functions, just introduced.

2.1. $\mathcal{P}$-polynions Algebra

Consider now $w = v_2$ in (13), where v_2 is the element of the cyclical basis (7) of $\mathcal{P}$-polynions algebra. Writting $\varphi(xv_2) = \sum_{i=1}^{4} \varphi_i(x)\, v_i$ and utilising (12) and (13_1), we get $\varphi_1(x) = \frac{1}{2}(\cosh 2x + \cos 2x)$, $\varphi_2(x) = \varphi_3(x) = \varphi_4(x)$ and so on for the other three functions.

Finally we obtain

$$(15)\quad \begin{cases} \varphi(xv_2) = \frac{1}{2}(\cosh 2x + \cos 2x)\, v_1, & \psi(xv_2) = \frac{1}{4}(\sinh 2x + \sin 2x)\, v_2, \\ \theta(xv_2) = \frac{1}{8}(\cosh 2x - \cos 2x)\, v_3, & \chi(xv_2) = \frac{1}{16}(\sinh 2x - \sin 2x)\, v_4. \end{cases}$$

If we utilise relations (14) and (9), then

$$(16)\quad \begin{cases} \varphi(xv_2) = [\varphi^2(x) + 4\theta^2(x)]\, v_1, & \psi(xv_2) = \frac{1}{2}[\varphi(x)\psi(x) + 4\theta(x)\chi(x)]\, v_2, \\ \theta(xv_2) = \frac{1}{2}[\psi^2(x) + 4\chi^2(x)]\, v_3, & \chi(xv_2) = \frac{1}{2}[\psi(x)\theta(x) - \varphi(x)\,\chi(x)]\, v_4. \end{cases}$$

Comparing (15) and (16) it results four relations between the usual Kryloff functions (5) and certain real trigonometric and hyperbolic functions. They have been obtained by other means in [3, (14)].

By continuing in the same way for elements v_3 and v_4, we can obtain

$$(17)\quad \begin{cases} \varphi(xv_3) = \varphi(2x)\, v_1, & \psi(xv_3) = \frac{1}{2}\psi(2x)\, v_3, \\ \theta(xv_3) = -\,\theta(2x)\, v_1, & \chi(xv_3) = -\frac{1}{2}\chi(2x)\, v_3, \end{cases}$$

and

$$(18)\quad \begin{cases} \varphi(xv_4) = \varphi(2xv_2), & \psi(xv_4) = \frac{1}{2}\psi(2xv_2), \\ \theta(xv_4) = -\,\theta(2xv_2), & \chi(xv_4) = 2\chi(2xv_2). \end{cases}$$

Now that Kryloff functions are defined for an element w, we may write xw instead of x in any of the relations (6), (8), (9) ... (11) or (15) ... (18).

For instance, if we introduce succesively in (6) xw_1 and yw_2, utilise the fundamental property of the exponential function and perform the necessary multiplications according to (6), it results

$$
(19)\quad \begin{cases}
\varphi(xw_1 + yw_2) = \varphi(xw_1)\,\varphi(yw_2) - 4\psi(xw_1)\,\chi(yw_2) - 4\theta(xw_1)\,\theta(yw_2) - \\
\qquad\qquad - 4\chi(xw_1)\,\psi(yw_2), \\
\psi(xw_1 + yw_2) = \varphi(xw_1)\,\psi(yw_2) + \psi(xw_1)\varphi(yw_2) - 4\theta(xw_1)\,\chi(yw_2) - \\
\qquad\qquad - 4\chi(xw_1)\,(\theta)yw_2), \\
\theta(xw_1 + yw_2) = \varphi(xw_1)\,\theta(yw_2) + \psi(xw_1)\,\psi(yw_2) + \theta(xw_1)\,\varphi(yw_2) - \\
\qquad\qquad - 4\chi(xw_1)\,\chi(yw_2), \\
\chi(xw_1 + yw_2) = \varphi(xw_1)\,\chi(yw_2) + \psi(xw_1)\,\theta(yw_2) + \theta(xw_1)\,\psi(yw_2) + \\
\qquad\qquad + \chi(xw_1)\,\varphi(yw_2).
\end{cases}
$$

Relations (19) allow to obtain easily a Kryloff function for an element of the $\mathcal{P}$— polynion algebra, expressed in an arbitrary basis, particularly (7), if we know the Kryloff functions of the basis elements.

If we introduce xv_2 instead of x in (6), then taking into account (16) and (8_1) (since $xv_2v_2 = xv_3$) we obtain a lot of relations involving the usual Kryloff functions, given also in [3, (13)].

Finally, we shall remark that (17) can be obtain from (15), if $x \to xv_2$, and similarly (18) from (17), by substituting $x \to xv_3$.

2.2. Sobrero Algebra

In [2] we have established that if one considers the cyclical basis in Sobrero algebra,

$$
(20)\qquad u^0 = u_1,\quad u = u_2,\quad u^2 = u_3,\quad u^3 = u_4 \quad \text{and} \quad u^4 = -2u^2 - u,
$$

then the trigonometric and hyperbolic functions for element v_2 are

$$
(21)\quad \begin{cases}
\cos(xu_2) = A(x)\,u_1 + B(x)\,u_3, & \cosh(xu_2) = \Phi(x)\,u_1 + \Theta(x)\,u_3, \\
\sin(xu_2) = C(x)\,u_2 + D(x)\,u_4, & \sinh(xu_2) = \Psi(x)\,u_2 + \Gamma(x)\,u_4.
\end{cases}
$$

Here

$$
(22)\quad \begin{cases}
A(x) = \dfrac{1}{2}\,(2\cosh x - x\sinh x), & B(x) = -\dfrac{1}{2}\,x\sinh x, \\[2ex]
C(x) = \dfrac{1}{2}\,(3\sinh x - x\cosh x), & D(x) = \dfrac{1}{2}\,(\sinh x - x\cosh x),
\end{cases}
$$

and

$$
(23)\quad \begin{cases}
\Phi(x) = \dfrac{1}{2}\,(2\cos x + x\sin x), & \Psi(x) = \dfrac{1}{2}\,(3\sin x - x\cos x), \\[2ex]
\Theta(x) = \dfrac{1}{2}\,x\sin x, & \Gamma(x) = \dfrac{1}{2}\,(\sin x - x\cos x),
\end{cases}
$$

the last functions being called Sobrero functions.

We can obtain now the Kryloff functions on Sobrero algebra by utilising the relations (14) and (21) ... (23). It results finally

$$
(24)\quad \begin{cases}
\varphi(xu_2) = [\varphi(x) + 2x\chi(x)]\,u_1 + 2x\chi(x)\,u_3, \\
\psi(xu_2) = \frac{1}{2}[3\psi(x) - x\varphi(x)]\,u_2 + \frac{1}{2}[\psi(x) - x\varphi(x)]\,u_4, \\
\theta(xu_2) = \frac{1}{2}[x\psi(x) - 2\theta(x)]\,u_1 + \frac{1}{2}x\psi(x)\,u_3, \\
\chi(xu_2) = \frac{1}{2}[x\theta(x) - 3\chi(x)]\,u_2 + \frac{1}{2}[x\theta(x) - \chi(x)]\,u_4,
\end{cases}
$$

where $\varphi(x)$, $\psi(x)$, $\theta(x)$, $\chi(x)$ are the usual Kryloff functions, given by (5).

Relations (24) can be simplified by means of the change of basis [3]

$$
(25)\qquad u_1 = \varepsilon_1,\quad u_2 = 2\varepsilon_2 + \varepsilon_3,\quad u_3 = -\varepsilon_1 + 4\varepsilon_4,\quad u_4 = -6\varepsilon_2 - \varepsilon_3,
$$

which leads to a multiplication table revealing that Sobrero algebra is the cartesian product of complex and dual algebras. Now

$$
(26)\quad \begin{cases}
\varphi(xu_2) = \varphi(x)\,\varepsilon_1 + 8x\chi(x)\,\varepsilon_4, & \psi(xu_2) = 2x\varphi(x)\,\varepsilon_2 + \psi(x)\,\varepsilon_3, \\
\theta(xu_2) = -\theta(x)\,\varepsilon_1 + 8x\psi(x)\,\varepsilon_4, & \chi(xu_2) = -2x\theta(x)\,\varepsilon_2 - \chi(x)\,\varepsilon_3.
\end{cases}
$$

Let be replaced in (26) x by xu_2; it results

$$
(27)\quad \begin{cases}
\varphi(xu_3) = \varphi(x)\,\varepsilon_1 + 16x\chi(x)\,\varepsilon_4, & \psi(xu_3) = 2x\varphi(x)\,\varepsilon_2 + \psi(x)\,\varepsilon_3, \\
\theta(xu_3) = \theta(x)\,\varepsilon_1 - 16x\psi(x)\,\varepsilon_4, & \chi(xu_3) = -2x\theta(x)\,\varepsilon_2 - \chi(x)\,\varepsilon_3.
\end{cases}
$$

By repeating the same operation starting from (27), we get

$$
(28)\quad \begin{cases}
\varphi(xu_4) = \varphi(x)\,\varepsilon_1 + 24x\chi(x)\,\varepsilon_4, & \psi(xu_4) = -6x\varphi(x)\,\varepsilon_2 - \psi(x)\,\varepsilon_3, \\
\theta(xu_4) = -\theta(x)\,\varepsilon_1 + 24x\psi(x)\,\varepsilon_4, & \chi(xu_4) = 6x\theta(x)\,\varepsilon_2 + \chi(x)\,\varepsilon_3.
\end{cases}
$$

Kryloff functions for the elements of the new basis (25) result directly from (14), taking into account the respective form of the involved trigonometric and hyperbolic functions. They are

$$
(29)\quad \begin{cases}
\varphi(x\varepsilon_1) = \varphi(x)\,\varepsilon_1, & \varphi(x\varepsilon_2) = \varepsilon_1, & \varphi(x\varepsilon_3) = \varphi(x)\,\varepsilon_1, & \varphi(x\varepsilon_4) = \varepsilon_1, \\
\psi(x\varepsilon_1) = \psi(x)\,\varepsilon_1, & \psi(x\varepsilon_2) = x\varepsilon_2, & \psi(x\varepsilon_3) = \psi(x)\,\varepsilon_3, & \psi(x\varepsilon_4) = x\varepsilon_4, \\
\theta(x\varepsilon_1) = \theta(x)\,\varepsilon_1, & \theta(x\varepsilon_2) = 0, & \theta(x\varepsilon_3) = -\theta(x)\,\varepsilon_1, & \theta(x\varepsilon_4) = 0, \\
\chi(x\varepsilon_1) = \chi(x)\,\varepsilon_1; & \chi(x\varepsilon_2) = 0; & \chi(x\varepsilon_3) = -\chi(x)\,\varepsilon_3; & \chi(x\varepsilon_4) = 0.
\end{cases}
$$

For an element of Sobrero algebra, expressed either in basis (20) or (25), the corresponding Kryloff functions may be obtained by applying (19).

3. Conclusion

1. Functions on associative, commutative algebras, with unit, are introduced by means of differential equations on these algebras. They can

be easily obtained if the solutions of some equations, with constant coefficients, on the real field, are beforehand known.

2. Kryloff functions of an algebra A_n are called the solutions of (12), satisfying the initial conditiond (13). They have the form (14) and verify relations (19).

3. For the elements of the cyclical basis of $\mathcal{P}$-polynions algebra and Sobrero algebra, Kryloff functions are respectively (15)... (18) and (26)...(29).

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FUNCŢII KRÎLOV PE ALGEBRE

(Rezumat)

Se introduc funcţii pe algebre asociative, comutative şi cu unitate, pe corpul real utilizînd ecuaţii diferenţiale pe aceste algebre. Ele se pot obţine cu uşurinţă dacă se cu nosc soluţiile anumitor ecuaţii diferenţiale ordinare, cu coeficienţi constanţi.

Se denumesc funcţii Krîlov pe o algebră A_n soluţiile ecuaţiei (12), care satisfac condiţiile iniţiale (13). Ele au în general aspectul (14) şi verifică relaţiile (19). Pentru elementele bazelor ciclice ale algebrei polinionilor $\mathcal{P}$ (7) şi algebrei lui Sobrero (25), funcţiile Krîlov corespunzătoare sînt date de (15) ... (18) şi respectiv (26) ... (29).

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MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

C. C. 517.91

A DIFFERENTIAL EQUATION WITH A SET MAPPING AS UNKNOWN

BY

DAN BORŞ

1. Let M be a metric space with the metric d, X an open subspace of M and T a tribe of subsets of M such that $X \in T$.

1.1. Let f be a continuous and bounded (L' being the constant of boundedness) mapping which satisfies the Lipschitz condition in the second argument (L being the Lipschitz constant) and applies an open subset D of $M \times R^1$ into R^1 (the set of real numbers).

1.2. Let (X, T, μ) a measure space and $\mu : T \to R^1$ be a positive measure such that $\mu(X) < L^{-1}$.

1.3. Let $\varphi : T \to R^1$ be a countably additive mapping with the property : $\forall \varepsilon > 0$, $\forall A, B \in T$, $\exists \eta > 0$ such that

$$\mu(A \setminus B) < \eta \quad \text{and} \quad \mu(B \setminus A) < \eta \tag{1}$$

implies

$$|\varphi(A \setminus B)| + |\varphi(B \setminus A)| < \varepsilon ;$$

if $\mu(A) = \mu(B) = 0$ we add to (1) the condition that $A \setminus B$ and $B \setminus A$ be enclosed into an open ball which radius equals η.

1.4. Suppose there exists a bijection $F : X \to T'$, T' being a subset of T, denoted by $F(x) = A_x$, $\forall x \in X$, with the property : $\forall \varepsilon > 0$, $\forall x \in X$, $\exists \eta > 0$ such that $\forall y \in X$,

$$d(x, y) < \eta \tag{2}$$

implies

$$\mu(A_x \setminus A_y) < \varepsilon \quad \text{and} \quad \mu(A_y \setminus A_x) < \varepsilon ;$$

if $\mu(A_x) = \mu(A_y) = 0$ we add to (2) the condition that $A_x \setminus A_y$ and $A_y \setminus A_x$ can be enclosed into an open ball which radius equals η.

1.5. Let $u : X \to R^1$ be a mapping defined by the relation $u = \varphi / T' \circ F$; we denote the restriction of φ to T' also by φ; then $u = \varphi \circ F$ or $u(x) = \varphi(A_x)$, $\forall x \in X$.

Under the conditions 1.1. ... 1.5. the mapping u is continuous [1].

2. The mapping φ has the unique Lebesgue decomposition $\varphi = \alpha + \sigma$ with $\alpha : T \to R^1$ an absolutely continuous mapping with respect to the measure μ and $\sigma : T \to R^1$ a singular mapping with respect to the measure μ; moreover, there exists a μ-integrable mapping $g : X \to R^1$ defined μ-almost everywhere in X and such that

$$\alpha(A) = \int_A g(x)\,\mathrm{d}\mu, \quad \forall x \in A, \quad \forall A \in T;$$

$\sigma(A) = \varphi(A \cap A_0)$ is a singular mapping with respect to the measure μ where A_0 is a set of zero measure that is an element of the ideal which contains all sets of zero measure belonging in T.

We have

$$\varphi(A) = \int_A g(x)\,\mathrm{d}\mu + \varphi(A \cap A_0), \quad \forall A \in T, \quad \forall x \in A \tag{3}$$

and this relation is also true for $A = A_x$ of T'. By 1.1., if D' is an open set of R^1 we take $D = X \times D'$ and $g(x) = f(x, \varphi(A_x))$ that is (3) becomes

$$\varphi(A_x) = \varphi(A_x \cap A_0) + \int_{A_x} f(y, \varphi(A_y))\,\mathrm{d}\mu \tag{3'}$$

for any $A_x \in T'$ and $y \in A_x$.

The mapping $\varphi' : x \to f(x, \varphi(A_x))$ is called the derivative of φ in $x \in X$ and the relation

$$\varphi'(x) = f(x, \varphi(A_x)) \tag{4}$$

for any $x \in X$ and $A_x \in T'$, is called a differential equation with set mapping φ as unknown; (3′) and (4) are assumed to be equivalent.

According to (3′) and (4), the equation $\varphi'(x) = 0$, $\forall x \in X$, has as solution the restriction to T' of the unique singular mapping $\sigma : T \to R^1$ with respect to the measure μ, defined by $\sigma(A) = \varphi(A \cap A_0)$, $\forall A \in T$.

3. The existence and unicity theorem given in [1] for the solutions of the equation (4) say that under the conditions 1.1. ... 1.5. there exists an unique mapping $\varphi : T \to R^1$ such that its restriction to T' satisfies (4) and has arbitrary given values in any intersection of sets of T' with a given zero measure set.

The set $T \times R^1$ is a quasivector space [2], [3] over R^1; the graph of a solution $\varphi : T' \to R^1$ of (4) is a subset of quasivector space $T \times R^1$ and its elements are $(A, \varphi(A))$, $\forall A \in T'$. The set $T \cap A_0$, for fixed A_0

in the ideal of zero measure sets, is a subtribe of T that is $(T \cap A_0) \times R^1$ is a nonbanal quasivector subspace of $T \times R^1$. The graph of a solution of (4) contains the subset of elements $(A \cap A_0, \varphi(A \cap A_0))$, $\forall A \in T'$, which belong to $(T \cap A_0) \times R^1$.

The values of φ in $A_x \cap A_0 \in T' \cap A_0$, $\forall x \in X$, depend only on x and we indicate them as we wish; it means that there exists a mapping $u_0 : X \to R^1$ defined by the relation $u_0(x) = \varphi(A_x \cap A_0)$, $\forall x \in X$, which is supposed continuous and bounded in X. The equation (3′) becomes

$$u(x) = u_0(x) + \int_{A_x} f(y, u(y))\, d\mu, \quad \forall x \in X, \quad \forall y \in A_x,$$

where (1°) u is continuous, (2°) $u(x) = u_0(x)$ for any $x \in X$ such that $\mu(A_x) = 0$ and (3°) u is bounded since $| u(x) - u_0(x) | < L' L^{-1}$, $\forall x \in X$.

Let U be the set of all mappings $u : X \to R^1$ defined by the relation $u(x) = \varphi(A_x)$, $\forall x \in X$, satisfying the conditions 1°, 2°, 3°.

If we denote by Φ the mapping

$$(\Phi u)(x) = u_0(x) + \int_{A_x} f(y, u(y))\, d\mu$$

we prove that Φ applies U in U [1]. The set U of continuous and bounded mappings defined on the metric space X with values in the complete metric space R^1 becomes a complete metric space, [4, p. 40], with the metric

$$\delta(u, v) = \sup_{x \in X} | u(x) - v(x) |$$

for any $u, v \in U$. The mapping Φ is a contraction since

$$| (\Phi u)(x) - (\Phi v)(x) | \leqslant \int_{A_x} | f(y, u(y)) - f(y, v(y)) | \, d\mu \leqslant$$

$$\leqslant L \int_{A_x} | u(y) - v(y) | \, d\mu \leqslant L\mu(A_x)\ \delta(u, v) \leqslant L\mu(X)\ \ \delta(u, v),$$

where $0 \leqslant L\mu(X) < 1$ by 1.2. and there exists an unique mapping $u : X \to R^1$ such that $\Phi(u) = u$ that is a mapping $\varphi : T \to R^1$ such that φ / T' is unique and satisfies the equation (3′) or (4).

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O ECUAŢIE DIFERENŢIALĂ CU O FUNCŢIE DE MULŢIME CA NECUNOSCUTĂ

(Rezumat)

Se pune în evidenţă o legătură între rezultatele din [1] şi cele obţinute în [2], [3] şi se dă o altă demonstraţie a faptului că aplicaţia Φ este contractantă.

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D. K. 517.53

ÜBERDECKUNGSSATZ FÜR DIE KLASSE θ-SPIRALISCHER FUNKTIONEN

VON

I. M. GALPERIN

S a t z. *Es sollen n logaritmische Spiralen von gleicher Neigung* $a = \operatorname{tg}\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *und mit gemeinsamen asymptotischen Punkt* $w = 0$ *jede Kreislinie* $\rho = \text{const}$ *in n Scheitelpunkten eines regelmässigen n-Eck's durchschneiden.*

Auf jeder Spirale sei M_j, $(j = 1, 2, 3, \ldots, n)$ — *ein Punkt der Kreislinie*

$$\rho = \exp\left\{-\frac{2}{n}[\cos\theta \ln|2\cos\theta| + \theta\sin\theta]\cos\theta\right\}.$$

Gehört die reguläre Funktion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

zu der Klasse S_θ *der θ-spiralischer Funktionen, d, h.,*

$$\mathcal{R}e\left(e^{-i\theta}\frac{zf'(z)}{f(z)}\right) > 0, \quad |z| < 1,$$

so enthält die Gestalt G des Kreises $|z| < 1$ *bei der Funktionenabbildung* $w = f(z) \in S_\theta$ *mindestens einen der Punkte* M_j, $(j = 1, 2, 3, \ldots, n)$ *und zugleich den Bogen der entsprechenden Spirale von* $w = 0$ *bis zu diesem Punkte.*

B e w e i s. Es sei C_j — ein Bogen j-er der genannten Spiralen vom Punkte M_j bis ins Unendliche und Q — längs allen Kurven C_j geschlitzte w-Ebene.

Mit $M_j \overline{\in} G$ ist es auch $C_j \overline{\subset} G$. Die Funktion

$$w = f_0(z) = z(1 - z^n e^{-in\alpha})^{-2\sigma} \equiv R e^{i\Phi}, \quad z = re^{i\varphi},$$

wo $\sigma = \cos\theta\, e^{i\theta}$ und α — ein entsprechender Weise gewählter konstanter Winkel ist, bildet die Schlitzebene Q auf den Einheitskreis $|z| < 1$ ab. Daher $G \subset Q$, obwohl $f(z)$ und $f_0(z)$ die gleiche Normierung haben:

$$f(0) = f_0(0) = 0\,; \quad f'(0) = f_0'(0) = 1.$$

Das ist unmöglich.

Bei $r = 1$ bekomt man:

$$R = e^{\lambda}, \quad \lambda = -\frac{2}{n}\left[\sigma_1 \ln\left|2\sin\frac{n\vartheta}{2}\right| - \sigma_2\left(\frac{n\vartheta}{2} \pm \frac{\pi}{2}\right)\right],$$

$$\Phi = \vartheta + \alpha - \frac{2}{n}\left[\sigma_2 \ln\left|2\sin\frac{n\vartheta}{2}\right| + \sigma_1\left(\frac{n\vartheta}{2} \pm \frac{\pi}{2}\right)\right];$$

$$\sigma_1 = \cos^2\theta, \quad \sigma_2 = \sin\theta\cos\theta, \quad \vartheta = \varphi - \alpha$$

den Pluszeichen wählen wir bei $\sin\frac{n\vartheta}{2} > 0$, den Minuszeichen — bei $\sin\frac{n\vartheta}{2} < 0$.

Es ist leicht nachzuprüfen, dass

$$\operatorname{tg}\varphi = R\frac{d\Phi}{dR} = \frac{d\Phi}{d\lambda} = \operatorname{tg}\theta, \quad (\varphi = \theta)\,;$$

$$\lambda_0 = \min\lambda = -\frac{2}{n}\left[\sigma_1 \ln|2\cos\theta| + \sigma_2\theta\right], \quad 0 < n\vartheta < 2\pi.$$

Ist $\theta = 0$; so bekomt man entsprechenden Satz für die Klasse der sternförmigen Funktionen, der zu allererst von Strohhäcker [1] bewiesen war.

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Kijewer Institut für Kraftwagen und Strassenbauwesen, UdSSR

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TEOREMĂ DE ACOPERIRE PENTRU CLASA FUNCŢIILOR θ-SPIRALE

(Rezumat)

Se extinde la clasa funcţiilor θ-spirale o teoremă de acoperire dată de E. Strohhäcker pentru funcţiile stelate.

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D. C, 517,86

ON CERTAIN INFINITE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

BY

R. Y. DENIS

1. Recently, Sharma [4] evaluted an integral involving Meijer's G-function under different set of conditions which includes a result due to MacRobert [3]. In the present paper we discuss the evaluation of an integral involving G-function of two variables (cf. Agarwal [1]) under different set of conditions, which includes the above result due to Sharma [4] as its special case.

2. We, now proceed to prove that

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix} a(x+y)^{-m}x^{m-r} \\ b(x+y)^{-m}x^{m-r}\end{matrix}\middle|\begin{matrix}(\varepsilon_p)\\ (\gamma_t);(\gamma'_{t'})\\ (\delta_s)\\ (\beta_q);(\beta'_{q'})\end{matrix}\right] dx =$$

$$= \sqrt{2\pi}\,\frac{(m-r)^{\beta-1/2}\,y^{-\alpha}\,r^{\alpha-1/2}}{m^{\alpha+\beta-1/2}}\times \tag{2.1}$$

$$\times G^{n+m,\nu_1,\nu_2,m_1,m_2}_{p+m,[t:t'],s+m,[q:q']}\left[\begin{matrix}\dfrac{a(m-r)^{m-r}r^r}{m^m y^r}\\[2ex] \dfrac{b(m-r)^{m-r}r^r}{m^m y^r}\end{matrix}\middle|\begin{matrix}\Delta[m-r;1-\beta],\ \Delta[r;1-\alpha],\ (\varepsilon_p)\\ (\gamma_t);(\gamma'_{t'})\\ (\delta_s),\ \Delta[m;\alpha+\beta]\\ (\beta_q);(\beta'_{q'})\end{matrix}\right],$$

where G-function appearing in (2.1) is G-function of two variables (cf. Agarwal [1]), r and m are positive integers and $m > r$; (It is not necessary to have equal number of parameters γ and γ' and so also β and β'). The symbol (α_p) used in (2.1) stands for the sequence of parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\Delta[p;\alpha]$ denotes the parameters

$$\frac{\alpha}{p}, \frac{\alpha+1}{p}, \ldots, \frac{\alpha+p-1}{p}.$$

Equation (2.1) holds when

$$0 \leqslant n \leqslant p, \quad 0 \leqslant \nu_1 \leqslant t, \quad 0 \leqslant \nu_2 \leqslant t', \quad 0 \leqslant m_1 \leqslant q, \quad 0 \leqslant m_2 \leqslant q',$$

$$p + s + q + t < 2(n + m_1 + \nu_1), \quad p + q' + s + t' < 2(n + m_2 + \nu_2)$$

and

$$|\arg a| < \pi \left[n + m_1 + \nu_1 - \frac{p+q+s+t}{2} \right],$$

$$|\arg b| < \pi \left[n + m_2 + \nu_2 - \frac{p+q'+s+t'}{2} \right],$$

$$\left| \arg \frac{a(m-r)^{m-r} r^r}{m^m y^r} \right| < \pi \left[n + m_1 + \nu_1 - \frac{p+q+s+t}{2} \right],$$

$$\left| \arg \frac{b(m-r)^{m-r} r^r}{m^m y^r} \right| < \pi \left[n + m_2 + \nu_2 - \frac{p+q'+s+t'}{2} \right],$$

$$\mathrm{Rl}(\beta) > \mathrm{Rl}(\alpha) > 0, \quad \mathrm{Rl}\,[\beta + (m-r)\{(\beta_{m_1}) + (\beta'_{m_2})\}] > 0,$$

$$t < q, \ \nu_1 \geqslant 1, \ t + q < 2(m_1 + \nu_1), \qquad t' < q', \ \nu_2 \geqslant 1, \ t' + q' < 2(m_2 + \nu_2),$$

$$|\arg a| < \pi \left[m_1 + \nu_1 - \frac{t+q}{2} \right], \qquad |\arg b| < \pi \left[m_2 + \nu_2 - \frac{t'+q'}{2} \right],$$

$$\mathrm{Rl}\,[mu + rv + \alpha] > 0,$$

where u and v are some finite functions of the parameters (γ_t), $(\gamma'_{t'})$, (β_q) and $(\beta'_{q'})$.

To prove (2.1), we replace the G-function on the left of it by equivalent double integral (cf. Agarwal [1]), change the order of integration and evaluate the innermost integral from 0 to ∞, with the help of the following result

$$(2.2) \qquad \int_0^\infty x^{\alpha-1}(x+y)^{-\beta}\,dx = \Gamma\left[\begin{matrix}\alpha,\ \beta-\alpha\,; \\ \beta\end{matrix}\right] y^{\alpha-\beta},$$

for

$$\mathrm{Rl}\,(\beta) > \mathrm{Rl}\,(\alpha) > 0.$$

After some simplification replace the double integral by its equivalent G-function of two variables, and this completes the proof of (2.1).

3. In this section we shall discuss few of the special cases of the main result (2.1).

If in (2.1) we take $p = 0 = s = b = \beta_1$, $t = t'$, $q = q'$, $q \geqslant t$, $m_2 = 1$ and then replace the $G(x)$ functions on both sides by their equivalent $G(1/x)$ functions, we get

$$(3.1) \qquad \int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}\, G^{\nu_1,m_1}_{q,t}\left[z(x+y)^m x^{r-m}\,\middle|\,\begin{matrix}1-(\beta_q) \\ (\gamma_t)\end{matrix}\right] dx =$$

$$= \sqrt{2\pi}\,\frac{(m-r)^{\beta-1/2}\, r^{\alpha-1/2}}{m^{\alpha+\beta-1/2}\, y^\alpha}\, G^{m+\nu_1,m_1}_{m+q,t+m}\left[zy^r\frac{m^m(m-r)^{r-m}}{r^r}\,\middle|\,\begin{matrix}1-(\beta_q),\Delta[m\,;\alpha+\beta] \\ \Delta[m-r;\,\beta],\Delta[r;\alpha],(\gamma_t)\end{matrix}\right],$$

where r and m are positive integers $(m > r)$, $z = 1/a$,

$$0 \leqslant \nu_1 \leqslant t, \quad 0 \leqslant m_1 \leqslant q, \quad t + q < 2(m_1 + \nu_1),$$

$$|\arg zy^r| < \pi\left[m_1 + \nu_1 - \frac{t+q}{2}\right].$$

The conditions $q = q$, $t = t$ and $q \geqslant t$ can now be removed by an appeal to analytic continuation.

Relation (3.1) is a result due to Sharma [4, (7)].

It may be remarked here that the left side of (2.1) can be evaluated when $r > m$ (r and m being positive integers) and one could easily get another result due to Sharma [4, (8)] as its special case. It is evident that all the results due to Sharma [4] which are generalizations of the results due to MacRobert [3], can be deduced as the special cases of the results discussed in this paper.

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ASUPRA UNOR INTEGRALE INFINITE CONŢINÎND FUNCŢIA HIPERGEOMETRICĂ GENERALIZATĂ

(Rezumat)

Se evaluează în condiţii mai generale o integrală conţinînd funcţia G în două variabile [1], obţinînd rezultate care extind pe cele ale lui Sharma [4] şi MacRobert [3]

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Secţia I
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

D. K. 517.912

ÜBER EINE NICHTLINEARE DIFFERENTIALGLEICHUNG BELIEBIGER ORDNUNG

VON

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Herrn WOLFGANG HAHN zum sechzigsten Geburtstag gewidmet

1. Bekanntlich besitzt die lineare Differentialgleichung

$$L_n[x] \equiv x^{(n)} + a_1 x^{(n-1)} + \dots + a_n x = p(t), \quad n \geqslant 1$$

bei einer beliebigen (stetigen) rechten Seite $p(t) \equiv p(t+T)$ genau eine T-periodische Lösung $x(t) = Up(t)$, falls die charakteristische Gleichung

$$D_n(\lambda) \equiv \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$$

keine Wurzel $\lambda = ir\omega$ (i—imaginäre Einheit, r—nicht-negative ganze Zahl, $\omega = 2\pi/T$) hat ; diese Lösung gestattet überdies die Abschätzung

$$\operatorname*{Max}_{[0,T]} |x(t)| = \| x(t) \| \leqslant k \| p(t) \|,$$

wobei $k = k(a_1, \dots, a_n)$.

2. Tritt ausserdem ein (stetiges) nichtlineares Zusatzglied auf, d.h.

$$L_n[x] + \varepsilon g(x) = p(t), \tag{1}$$

so kann man leicht zeigen, dass bei kleinem $|\varepsilon|$ ebenfalls eine T-periodische Lösung vorhanden ist.

Man bildet den Banach-Raum B der über $[0, T]$ stetigen Funktioonen $u(t)$ mit der Randbedingung $u(0) = u(T)$ und verwendet die Norm

$$\| u(t) \| = \operatorname*{Max}_{[0,T]} | u(t) |.$$

In B betrachtet man die (beschränkte, stetige, kompakte) Abbildung

$$u(t) \to U_\varepsilon u(t) = Up(t) - \varepsilon Ug(u(t)).$$

Man setzt

$$\gamma(r) = \operatorname*{Max}_{x \leqslant r} | g(x) |, \quad m = \| p(t) \|, \quad R \geqslant 2km ;$$

ferner schreibt man vor:

$$| \varepsilon | \leqslant \varepsilon_0 \leqslant \frac{R}{2k\gamma(R)} .$$

Im Fall

$$(*) \qquad \liminf_{r \to \infty} \frac{\gamma(r)}{r} = 0$$

kann man offenbar ε_0 beliebig wählen.

Dann wird für $\| u(t) \| \leqslant R$ (Sphäre $S_R \subset B$)

$$\| U_\varepsilon u(t) \| \leqslant km + | \varepsilon | k\gamma(R) \leqslant \frac{R}{2} + \varepsilon_0 k\gamma(R) \leqslant R.$$

Das bedeutet $U_\varepsilon S_R \subset S_R$, und nach dem Fixpunktsatz von Schauder liegt in S_R wenigstens ein Fixpunkt

$$x(t) = U_\varepsilon x(t),$$

der eine T-periodische Lösung von (1) darstellt.

3. Wir wollen zeigen, dass dieses Ergebnis unter gewissen Bedingungen erhalten bleibt, wenn der Koeffizient $a_n = 0$ ist; dann ist die Randwertaufgabe $x(0) = x(X)$ bei der linearisierten Differentialgleichung ($\varepsilon = 0$) nur noch bedingt—und nicht mehr eindeutig—lösbar. Dazu verwenden wir den Fixpunktsatz von Leray-Schauder, bezeichnen aber die Ordnung der Differentialgleichung mit $n + 1$ und trennen die Fälle $n \geqslant 1$ und $n = 0$. Es wird behauptet:

Satz 1. *Die Differentialgleichung*

$$(2) \qquad L_n[x'] + \varepsilon g(x) \equiv x^{(n+1)} + a_1 x^{(n)} + \ldots + a_n x' + \varepsilon g(x) = p(t), \quad n \geqslant 1,$$

in der $p(t)$ und $g(x)$ stetige Funktionen sein sollen, besitzt wenigstens eine T-periodische Losung, wenn

(i) $D_n(\lambda) \neq 0$ für $\lambda = \mathrm{i}\, r\omega$;

(ii) $\int_0^T p(t)\, \mathrm{d}t = 0$;

(iii) $xg(x) > 0 \quad (< 0)$ für $| x | \geqslant h$;

(iv) $|\varepsilon| \leqslant \varepsilon_0$.

Nach Voraussetzung (ii) ist die Funktion

$$P(t) = \int_0^t p(\tau)\, d\tau$$

periodisch mit der Periode T.

Zum Beweis des Satzes betrachten wir erst einmal die algebraische Gleichung $\Delta(\lambda, a) \equiv \gamma D_n(\lambda) + a = 0$, wobei $\Delta(0, a) = a$, $\Delta(\lambda, 0) = \lambda D_n(\lambda)$.

Wegen der stetigen Abhängigkeit der Wurzeln vom Koeffizienten a lässt sich erreichen, dass $\Delta(\lambda, a) \neq 0$ für $\lambda = ir\omega$, indem man $0 < |a|$ hinreichend klein wählt. Das Vorzeichen von a bestimmen wir so, dass $\varepsilon a x g(x) > 0$, $(|x| \geqslant h)$.

Bei den weiteren Überlegungen sei beispielsweise

$$\varepsilon x g(x) > 0 \quad (|x| \geqslant h), \quad \text{also} \quad a > 0.$$

Wir bilden die Differentialgleichung

$$L_n[x'] + ax = \mu(p(t) + ax - \varepsilon g(x)), \quad 0 \leqslant \mu \leqslant 1; \tag{3}$$

für $\mu = 0$ entsteht eine homogene lineare Gleichung mit der einzigen T-periodischen Lösung $x(t) \equiv 0$, und für $\mu = 1$ ergibt sich die Grundgleichung (2).

Die möglicherweise vorhandenen T-periodischen Lösungen von (3) kann man als Fixpunkte eines in $B = \{u(t)\}$ definierten (beschränkten, stetigen, kompakten) Operators $\mu V u(t)$ deuten. Nach Leray-Schauder gibt es zu jedem Wert $\mu \in [0, 1]$ wenigstens einen Fixpunkt, sofern die Menge der Fixpunkte über dem offenen Parameterintervall $(0, 1)$ beschränkt ist.

Um die a priori-Beschränktheit dieser periodischen Lösungen $x(t)$ von (3) nachzuweisen, betrachten wir eine solche und lassen dabei auch $\mu = 1$ zu. Für $y(t) = x'(t)$ besteht die Gleichung

$$\begin{aligned} L_n[y] &= q(t) \equiv \mu p(t) - g_\mu(x(t)), \\ g_\mu(x) &= (1 - \mu)ax + \mu\varepsilon g(x), \end{aligned} \tag{4}$$

aus der [mit $\| x(t) \| = r$] wie oben folgt:

$$\| y(t) \| \leqslant k \| q(t) \| \leqslant k(\mu m + (1 - \mu)ar + \mu |\varepsilon| \gamma(r)).$$

Statt (4) schreiben wir nun

$$\frac{d}{dt}(L_n[x] - \mu P(t)) = - g_\mu(x(t)); \tag{5}$$

wegen

$$\int_0^T g_\mu(x(t))\,dt = 0 \quad \text{und} \quad xg_\mu(x) > 0 \quad (\,|\,x\,| \geqq h)$$

ist $|\,x(t)\,| \geqslant h$ für $0 \leqslant t \leqslant T$ ausgeschlossen. Daher gibt es einen Wert $\tau \in [0, T]$, für den $|\,x(\tau)\,| < h$. Im Intervall $[\tau, \tau + T]$ gilt dann

$$|\,x(t)\,| \leqslant |\,x(\tau)\,| + T\,\|\,y\,\| < h + kT\,(\mu m + (1-\mu)\,ar +$$
$$+ \mu\,|\,\varepsilon\,|\,\gamma(r)) < h + kT(m + ar + |\,\varepsilon\,|\,\gamma(r)),$$

also auch

$$r < h + kT(m + ar + |\,\varepsilon\,|\,\gamma(r)). \tag{6}$$

Bei $\mu = 1$ erhält man sogar die Abschätzung

$$\|\,x(t)\,\| = r < (h + kTm) + |\,\varepsilon\,|\,k\,T\gamma(r). \tag{7}$$

Denkt man sich in (6) $a < 1/(kT)$ gewählt, so kann man weiterrechnen

$$1 < \frac{h + kTm}{1 - akT}\,\frac{1}{r} + \frac{kT}{1 - akT}\,|\,\varepsilon\,|\,\frac{\gamma(r)}{r}. \tag{8}$$

Im Falle (*) folgt aus (8) bei beliebigem ε, dass $r \leqslant R(a, k, T, \varepsilon)$ sein muss, und die a priori-Beschränktheit ist damit schon bewiesen.

Im Falle

$$\liminf_{r\to\infty} \frac{\gamma(r)}{r} > 0 \tag{**}$$

bestimmt man zu vorgegebenem $\vartheta \in (0, 1)$, $R = (h + kTm)/\vartheta$, sowie

$$\varepsilon_0 = \frac{1-\vartheta}{kT}\,\frac{R}{\gamma(R)}.$$

Damit ε_0 möglichst gross wird, geht man von einem solchen Wert ϑ aus, der die Bedingung

$$\frac{\vartheta}{1-\vartheta}\,\gamma\left(\frac{h + kTm}{\vartheta}\right) = \text{Min !}$$

erfüllt. Diese lässt sich im Beispiel $\gamma(r) = cr^\rho\,(\rho > 1)$ als $(1-\vartheta)\vartheta_\rho^{-1} = \text{Max !}$ schreiben und gestattet die Lösung $\vartheta = 1 - 1/\rho$.

Ändert man in der Grundgleichung (2) $g(x)$ in

$$g^*(x) = \begin{cases} g(x), & |\,x\,| \leqslant R \\ g(R\,\operatorname{sgn} x), & |\,x\,| \geqslant R \end{cases}$$

ab, so wird (bei entsprechender Bezeichnung)

$$\gamma^*(r) \leqslant \gamma(R) \quad \text{und} \quad \lim_{r\to\infty} \frac{\gamma^*(r)}{r} = 0.$$

Wie zuvor gezeigt wurde, ist daher eine T-periodische Lösung gesichert; ihre Norm lässt sich für $\mu = 1$ gemäss (7) abschätzen:

$$\| x(t) \| = r < (h + kTm) + | \varepsilon | kT\gamma^*(r). \tag{9}$$

Hieraus folgt für $| \varepsilon | \leqslant \varepsilon_0$ weiter $r < \vartheta R + \varepsilon_0 kT\gamma(R) = R$.

Macht man die Abänderung der Nichtlinearität rückgängig, so bleibt diese periodische Lösung offenbar erhalten.

In Sonderfall $n = 0$ gilt

S a t z 2. *Die Differentialgleichung*

$$x' + g(x) = p(t) \tag{10}$$

besitzt eine T-periodische Lösung, wenn

(i) $$xg(x) \geqslant 0 \ (\leqslant 0) \textit{ für } | x | \geqslant h;$$

(ii) $$\int_0^T p(t)\mathrm{d}t = 0.$$

Liegt in (i) der erste Fall vor, so bildet man mit einer positiven Konstante a die Gleichung

$$x' + ax = \mu(p(t) + ax - g(x)), \quad 0 \leqslant \mu \leqslant 1. \tag{11}$$

Zu einem Parameterwert $\mu \in (0, 1)$ existiere eine T-periodische Lösung $x(t)$; für sie gilt

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - \mu P(t)) = - g_\mu(x(t)),$$

also $\int_0^T g_\mu(x(t))\, \mathrm{d}t = 0$. Weil für $| x | \geqslant h$, $xg_\mu(x) \geqslant (1 - \mu)ax^2 > 0$ ist, können wir wiederum schliessen, dass $| x(\tau) | \leqslant h$ für gewisse Werte $\tau \in [0, T]$.

Hätte man einmal $x(t_1) > h$, so würde man setzen:

$$t_0 = \sup \{\tau : \tau < t_1, \quad x(\tau) = h\}.$$

Dann wäre

$$x(t_1) = x(t_0) + (t_1 - t_0)\, x'(t_1 + \vartheta(t_1 - t_0)) < h + Tm.$$

Analog liesse sich der Fall $x(t_1) < -h$ behandeln.

Insgesamt ergibt sich die a priori-Abschätzung $\| x(t) \| \leqslant h + Tm$, durch die der Satz bewiesen ist.

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ASUPRA UNEI ECUAŢII DIFERENŢIALE NELINIARE DE ORDIN OARECARE

(Rezumat)

Folosind o teoremă a lui Leray-Schauder, se demonstrează că ecuaţia diferenţială de ordin oarecare cu termen neliniar şi funcţie perturbatoare periodică (2) admite cel puţin o soluţie periodică în cazul cînd parametrul de pe lîngă termenul neliniar este suficient de mic şi unele condiţii foarte generale sînt satisfăcute.

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EVENTUAL UNIFORM ASYMPTOTIC STABILITY OF CONTROL SYSTEMS

BY

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1. Let J denote $0 \leqslant t < \infty$, R^n the Euclidean n-space, $S = [x \in R^n : \|x\| < \rho]$, $\rho > 0$, $\|x\|$ any convenient norm of $x \in R^n$ and R_+ the non-negative real line. We shall denote by $C[J \times R^m, R^n]$ the class of functions defined and continuous on $J \times R^m$ taking values in R^n, and k will denote the class of functions $b \in C[[0, \rho), R_+]$ such that $b(0) = 0$ and $b(y)$ is monotonic increasing in y.

2. Consider the control system

$$x' = f(t, x, u), \quad \left(' = \frac{d}{dt}\right), \tag{2.1}$$

where $f \in C[J \times S_\rho \times R^m, R^n]$ and u is a control vector. Let $u_0(t)$ be some control function continuous on J. Then the control system (2.1) reduces to

$$x' = f(t, x, u_0(t)). \tag{2.2}$$

The aim of this note is concerned with the eventual uniform asymptotic stability of the control system (2.1), which we define next.

D e f i n i t i o n 2.1. The set $x = 0$ of system (2.2) is said to be *eventual uniform asymptotically stable*, if the following two conditions hold:

a) for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ and a $\tau = \tau(\varepsilon) > 0$ such that the inequality $\|x_0\| \leqslant \delta$ implies

$$\|x(t, t_0, x_0)\| < \varepsilon, \qquad t \geqslant t_0 \geqslant \tau;$$

b) for every $\varepsilon > 0$, there exists positive numbers δ_0, τ_0 and $T(\varepsilon)$, such that, if $\|x_0\| \leqslant \delta_0$, and $t \geqslant t_0 + T(\varepsilon)$ where $t_0 \geqslant \tau_0$, then

$$\|x(t, t_0, x_0)\| < \varepsilon.$$

In the discussion we will be using the following theorem from our recent work; the proof can be found in [1].

Theorem 2.1. *Assume that*

(i) $V(t, x) \in C[J \times S_\rho, R_+]$, $V(t, x)$ *is locally Lipschitzian in* x *and*

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|)$$

for $0 < r < \| x \| < \rho$ *and* $t \geqslant \Theta(r)$, *where* $a, b \in k$ *and* $\Theta(y)$ *is continuous and decreasing in* y *for* $0 < y < \rho$;

(ii) $g \in C[J \times R_+ \times R_+, R]$, $g(t, y, z)$ *is non-decreasing in* z *for each* (t, y) *and the set* $y = 0$ of $y' = g(t, y, \lambda(t))$ *is eventually asymptotically stable*;

(iii) *for* $0 < r < \| x \| < \rho$ *and* $t \geqslant \Theta(r)$, $D^+V(t, x) \leqslant g(t, V(t, x), U(t, u))$, where $U \in C[J \times R^m, R_+]$.

Then the control system (2.1) *is eventually asymptotically stable.*

With respect to the aims of this note we state the following theorem.

3. Theorem 3.1. *Assume that*

(i) *the origin of system* (2.2) *is eventually uniformly asymptotically stable*

(ii) *for* $r \leqslant \| x \| < \rho$, $t \geqslant \Theta(r)$

$$\| f(t, x, u_1) - f(t, x, u_2) \| \leqslant L(r) \| u_1 - u_2) \|,$$

where $\Theta(r)$ *is non-increasing in* r;

(iii) $0 \leqslant \lambda(t)$ *is continuous for* $t \geqslant 0$ *and satisfies*

$$p(t) = \int_t^{t+1} \lambda(s) \mathrm{d}s \to 0 \quad as \quad t \to \infty.$$

Then under the above conditions the origin of the control system (2.1) *is eventually uniformly asymptotically stable for every control* $u = u(t) \in \Omega$ *where the set of all admissible controls is given by*

$$\Omega = \{u : \| u - u_0 \| \leqslant \lambda(t),\ t \in J\}.$$

Proof. By assumption (i) applying Wexler's theorem [2], there exists a Lyapunov function $V(t, x) \in C[J \times S_\rho, R_+]$, such that, for $0 < r \leqslant \| x \| < \rho$ and $t \geqslant \Theta(r)$, where $\Theta(r)$ is continuous for $0 \leqslant r < \rho$ and non-increasing in r,

A) $a(\| x \|) \leqslant V(t, x) \leqslant b(\| x \|)$,

B) $\| V(t, x) - V(t, y) \| \leqslant M(r) \| x - y \|$,

C) $$D^+V_{u_0}(t, x) = \lim_{h \to 0} \frac{1}{h} [V(t + h, x + h f(t, x, u_0(t))) - V(t, x)] \leqslant$$

$$\leqslant - C(\| x \|), \quad C \in k.$$

Let $u \in \Omega$, $r \leqslant \| x \| < \rho$ and $t \geqslant \Theta(r)$. Then

$$D^+ V_u(t, x) = \lim_{h \to 0} \frac{1}{h} [V(t+h, x + h f(t, x, u)) - V(t, x)] \leqslant$$

$$\leqslant D^+ V_{u_0}(t, x) + M(r) \| f(t, x, u) - f(t, x, u_0) \|,$$

becuase of the Lipschitzian character of $V(t, x)$. Applying assumption (ii) of the theorem and the fact that $D^+ V_{u_0}(t, x) \leqslant - C(\| x \|)$ we have

$$D^+ V_u(t, x) \leqslant - C(\| x \|) + M(r) L(r) \| u - u_0(t) \|,$$

or

$$D^+ V_u(t, x) \leqslant - C(\| x \|) + N\lambda(t), \quad \text{since} \quad u \in \Omega$$

when $N = M(r)L(r)$. Now, using (A) we can write

$$D^+ V_u(t, x) \leqslant - \beta(V(t, x)) + N\lambda(t),$$

where $\beta(\xi) = C(b^{-1}(\xi))$. Applying Theorem 2.1 we have

$$y' = - \beta(y) + N\lambda(t). \tag{3.1}$$

Equation (3.1) can be written as

$$y(t_2) = y(t_1) - \int_{t_1}^{t_2} \beta(y(s))\, \mathrm{d}s + N \int_{t_1}^{t_2} \lambda(s) \mathrm{d}s. \tag{3.2}$$

Observe that

$$\int_{t_0 - 1}^{t} \gamma(s) \mathrm{d}s = \int_{t_0 - 1}^{t} \left\{ \int_{s}^{s+1} p(u) \mathrm{d}u \right\} \mathrm{d}s \geqslant \int_{t_0}^{t} \left\{ \int_{u-1}^{u} p(u) \mathrm{d}s \right\} \mathrm{d}u = \int_{t_0}^{t} p(u) \mathrm{d}u \quad \text{for } t \geqslant t_0 \geqslant 1.$$

Hence, equation (3.2) becomes

$$y(t_2) \leqslant y(t_1) - \int_{t_1}^{t_2} \beta(y(s)) \mathrm{d}s + NQ(t_1)[t_2 - t_1 + 1], \tag{3.3}$$

where

$$Q(t) = \sup_{t-1 < s < \infty} p(s).$$

Note that $Q(t) \to 0$ as $t \to \infty$ in view of assumption (iii). Let $\varepsilon > 0$ be given and let $\delta = \dfrac{\varepsilon}{2}$. There exists a $\tau_0 > 0$ such that

$$NQ(\tau_0) < \frac{\varepsilon}{2} \quad \text{and} \quad 2NQ(\tau_0) < \beta(\delta). \tag{3.4}$$

Choose $t_0 \geqslant \tau_0 + 1$. We claim that with these δ, τ_0, $y=0$ of (3.1) is eventualy-uniformly stable. If not, there exists a $t_2 > t_1 \geqslant t_0 \geqslant \tau_0 + 1$ and a solul tion $y(t)$ of (3.1), such that

$$y(t_1) = \delta, \qquad y(t_2) = \varepsilon.$$

and

$$\delta < y(t) < \varepsilon, \qquad t \in (t_1, t_2). \tag{3.5}$$

Hence, in view of equation (3.3) we have

$$\varepsilon = y(t_2) \leqslant \frac{\varepsilon}{2} - \int_{t_1}^{t_2} \beta(y(s))\mathrm{d}s + NQ(t_1)\,[t_2 - t_1] + NQ(t_1).$$

Since $Q(t_0) \geqslant Q(t_1)$, it follows from inequality (3.5) that

$$\varepsilon \leqslant \frac{\varepsilon}{2} - \beta(\delta)(t_2 - t_1) + NQ(t_0)[t_2 - t_1] + NQ(t_0).$$

In view of inequalities (3.4) we obtain $\varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, a contradiction. This proves that the zero solution of (2.1) is eventually uniformly stable. To show that the solution is eventually uniformly asymptotically stable, let ε be fixed and let $0 < \eta < \varepsilon$. Choose $\delta(\eta) = \frac{\eta}{2}$, $\tau_0(\eta)$ as before. Also, choose

$$T = \frac{[\varepsilon + 2NQ(1) + (\beta\eta/2)\tau_0]}{\beta(\eta/2)}.$$

We claim that for some $t^* \in [t_0 + \tau_0, \tau_0 + T]$, we have $y(t^*) < \frac{\eta}{2}$. If this were not true, we would have

$$y(t) \geqslant \frac{\eta}{2}, \quad \text{for all} \quad t \in [t_0 + \tau_0,\ t_0 + \tau].$$

Then, as before we can write

$$0 < y(t_0 + T) \leqslant y_0 - \int_{t_0+\tau_0}^{t_0+T} \beta(y(s))\mathrm{d}s + NQ(t_0 + \tau_0)[T - \tau_0 + 1] =$$

$$= y_0 - \beta\left(\frac{\eta}{2}\right)[T - \tau_0] + NQ(t_0 + \tau_0)(T - \tau_0) + NQ(t_0 + \tau_0) =$$

$$= \frac{\varepsilon}{2} + \left[NQ(t_0 + \tau_0) - \beta\left(\frac{\eta}{2}\right)\right](T - \tau_0) + NQ(1).$$

The right hand side is equal to zero, because of (3.4) and the choice of T. This contradiction implies that

$$y(t^*) < \frac{\eta}{2} \quad \text{for} \quad t^* \geqslant t_0 + \tau_0.$$

Thus, in any case, $y(t) < \eta$ for $t \geqslant t_0 + T$. This proves that $y = 0$ is eventually uniformly asymptotically stable and the proof is complete.

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STABILITATEA ASIMPTOTICĂ UNIFORM EVENTUALĂ A SISTEMELOR DE CONTROL

(Rezumat)

Se defineşte noţiunea de stabilitate asimptotică uniform eventuală şi se stabilesc condiţiile în care soluţiile sistemului de control (2.2) prezintă acest tip de stabilitate. Se utilizează în acest scop o teoremă datorită lui D. Wexler [2].

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STABILITY OF HEREDITARY DIFFERENTIAL SYSTEMS [1])

BY

S. G. DEO

1. Introduction

Lyapunov's second method — one of the most versetile techniques in the theory of non-linear differential equations — has been employed continuously since its inception to solve a variety of problems such as existence, uniqueness, stability and boundedness of solutions. It was pointed out that a vector Lyapunov function is more advantageous in certain situations rather than a scalar function. This idea was exploited by Lakshmikantham [5] to define conditional stability and boundedness.

Recently, in an interesting paper Pieffer and Rouche [7] have applied the Lyapunov's second method to study partial stability behaviour of ordinary differential systems. Their article can be considered as a comprehensive work done, so far, on this topic. The object of the present paper is to extend this work in a number of directions.

Below we consider a differential system with hereditary character which describes a physical system in which the future behaviour of the system depends on its present state and also on its past history. During the past decade these systems have offered an exciting base for many mathematical researches. Out of a number of formulations introducing hereditary effect, the differential difference equations with constant time lag

$$\dot{x}(t) = F(t, x(t), x(t-\tau)) \tag{1.1}$$

has been studied in many details [1]. We consider below a more general system which includes (1.1) as a special case.

In order to extend the scope of the work of Pieffer and Rouche [7], we define some new stability concepts and introduce additional definitions of partial stabilities. The principle tool employed, to obtain

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the sufficient conditions for these stability properties to hold, is the Lyapunov's second method together with a differential inequality which includes the inequalities due to Wazewski [8], Lakshmikantham and Shendge [6] as special cases.

2. Notation

Let J denote the interval $0 \leqslant t < \infty$, R^n Euclidean n-space and C the space of continuous functions from $[-\tau, 0]$ into R^n, $\tau > 0$. For any $x \in R^n$ and $\varphi \in C$ define

$$\|x\| = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \|\varphi\|_c = \max_{-\tau \leqslant s < 0} \|\varphi(s)\|$$

respectively. Let R_H^n and C_H denote the sets $x \in R^n$ and $\varphi \in C$ for which $\|x\| \leqslant H$ and $\|\varphi\|_c \leqslant H$, $H > 0$ respectively. For any continuous function $x(\alpha)$ defined on $-\tau \leqslant \alpha < \infty$ and for any fixed $t \in J$

$$x(t+s) = (x_1(t+s),\ x_2(t+s), \ldots, x_n(t+s))$$

will denote the restriction of $x(\alpha)$ to the interval $[t-\tau, t]$. Let $\dot{x}(t)$ denote the right hand derivative of $x(\alpha)$ at $\alpha = t$. Now, consider the functional differential system

$$\dot{x}(t) = G(t, x(t), x(t+s)), \tag{2.1}$$

where $G(t, x, \varphi) \in R^n$ is defined and continuous on $J \times R_H^n \times C_H$. Let $x(t_0, \sigma)(t)$ be a solution of (2.1) with the initial condition $\sigma \in C_H$ at $t = t_0$, $t_0 \geqslant 0$. Assume that $G_i(t, \tilde{0}, \theta) = 0$ where $\tilde{0} \in R^n$ and $\theta \in C_H$ are the null elements and $i = 1, 2, \ldots, n$.

Observe that the equation (1.1) is a special case of (2.1) Equations of the form (2.1) are studied in [6]. More general systems of hereditary character are also studied in [3] however, we do not consider them here since the results, those follow, can be easily extended to such systems with suitable modifications.

In whatever follows, assume that the subscript i ranges over the integers 1 to n and for $0 \leqslant k \leqslant n$, the subscripts p and q range over the integers $1, 2, \ldots, k$ and $k+1, k+2, \ldots, n$ respectively. Further, suppose that $\|x\|_p = \sum_{j=1}^{k} |x_j|$ and $\|x\|_q = \sum_{j=k+1}^{n} |x_j|$ so that $\|x\| = \|x\|_p + \|x\|_q$.

Let M denote the class of functions $a(\alpha)$ which are defined, continuous and strictly monotone increasing on $0 \leqslant \alpha \leqslant H$ and $a(0) = 0$.

For any $\varphi \in C$ and $\psi \in C$ whenever we say that $\varphi \leqslant \psi$ we mean $\varphi_i(s) \leqslant \psi_i(s)$ for $-\tau \leqslant s \leqslant 0$.

3. Related Vector System

The „method of comparision" has been employed by many authors to study a variety of sophisticated problems in ordinary differential equations. This method crucially depends on a differential inequality which helps to compare the solution of a given system with the solution of the related ordinary differential system. Pieffer and Rouche [7] have used this method and vector Lyapunov function to prove Theorem 4 and have listed a number of references which include these types of results. The same idea has been used below, however, we compare the solutions of functional differential system with the known solutions of a similar type of system.

Now, in association with (2.1), consider the functional differential system

$$\dot{u}(t) = \mu(t, u, u(t+s)), \qquad u(t_0, \psi)(t_0) = \psi, \qquad t_0 \in J, \tag{3.1}$$

where the function $\mu(t, u, \varphi)$ is defined and continuous on $J \times R^n_H \times C_H$. Suppose that the function $\mu(t, u, \varphi)$ satisfies quasi-mixed monotone property defined below:

Definition. The function $\mu(t, u, \varphi)$ is said to satisfy a quasi-mixed monotone property if:

(i) $\mu_p(t, u, \varphi)$ is monotone increasing in u_j and φ_j, $(j = 1, 2, ..., k;$ $j \neq p)$ and monotone increasing in u_q and φ_q for each $t \in J$ and

(ii) $\mu_q(t, u, \varphi)$ is monotone decreasing in u_p and φ_p and monotone increasing in u_j and φ_j, $(j = k+1, k+2, ..., n; j \neq q$ for each $t \in J)$.

Observe that for $k = n$ and $k = 0$ the quasi-mixed monotone property reduces to quasi-monotone increasing and quasi-monotone decreasing property in u and φ respectively for each $t \in J$.

The „metod of comparision" in case of non-uniqueness of solutions, needs the existence of the maximal and minimal solutions of the related system [6], [7]. These solutions are extremal since they differ from other solutions in certain features. It is known that an extremal solution is extremal only with respect to that feature and need not be extremal with respect to the other features. We define below another type of extremal solution for the system (3.1).

Let $u(t_0, \psi)(t)$ be any solution of the differential system (3.1) existing to the right of t_0 such that for every solution $z(t_0, \psi)(t)$ the inequalities

$$\begin{cases} z_p(t_0, \psi)(t) \leqslant u_p(t_0, \psi)(t), \\ z_q(t_0, \psi)(t) \geqslant u_q(t_0, \psi)(t), \end{cases} \qquad (t \geqslant t_0)$$

are satisfied. Then $u(t_0, \psi)(t)$ is said to be the k max $(n-k)$ min solution of the system (3.1). We note that for $k = n$ and $k = 0$ this extremal solu-

tion reduces to the maximal and minimal solution respectively for the system (3.1). In general, except these two situations, maximal and minimal solutions for the system (3.1) will not exist.

The existence of k max $(n-k)$ min solution for the system (3.1) can be proved by introducing obvious changes in the arguments of the theorem 3 of [2]. We omit the details. The following example is illustrative.

Example. **Consider the following differential difference system**

$$\text{(3.2)} \qquad \begin{cases} \dot{x}_1(t) = -20[x_1^{3/5}(t-\tau) + 2x_2(t)], \\ \dot{x}_2(t) = -2x_1^{3/5}(t) - 3x_1^{2/5}(t-\tau), \end{cases} \quad (t > 0);$$

$$x_1(s) = x_2(s) = 0 \text{ for } -\tau \leqslant s \leqslant 0.$$

Note that the system possesses the null solution. The quasi-mixed monotonocity requirements stated in the definition also hold. Hence the system (3.2) has 1 max − 1 min solution and it is given by

$$x_1(t) = 32t^5, \qquad x^2(t) = -4[t^4 + (t-\tau)^3].$$

4. Vector Lyapunov Function

Let Z denote an open set in $J \times R_H^n$ and $v = [v_1, v_2, \ldots, v_n]$ be such that the function $v(t, x)$ is defined and continuous on Z and possesses the following properties:

$$\text{(4.1)} \qquad \begin{cases} \text{(i) } v \text{ satisfies local Lipschitz's condition on } \bar{Z}, \\ \text{(ii) } \|v(t, x)\|_p \to 0 \text{ as } \|x\| \to 0 \text{ for each } t \in J, \\ \text{(iii) } \|v(t, x)\| = 0 \text{ for all } (t, x) \in \bar{Z} - Z, \\ \text{(iv) } v_i(t, x) \text{ is positive and bounded on } Z. \end{cases}$$

Note that the derivative of the vector function v along the solutions of the system (2.1) is also a functional. Hence, the following definitions, for $(t, x) \in Z$ are in order:

$$\text{(4.2)} \qquad v_t = v(t+s, x(t+s)), \qquad (-\tau \leqslant s \leqslant 0),$$

$$\text{(4.3)} \qquad v^*(t, x(t), x(t+s)) = \limsup_{h \to 0+} \frac{1}{h}[v(t+h, x(t) + hG(t, x(t), x(t+s))) - v(t, x(t))].$$

The following lemma has been used as a basic tool in proving the main results and it is a generalization of a similar lemma proved in [6], [8].

L e m m a. *Let the function $v(t, x)$ satisfy the property* (4.1) *and the functional $v^*(t, x, \varphi)$ of* (4.3) *satisfy the inequalities for $(t, x) \in Z$*

$$(4.4)\qquad \begin{cases} v_p^*(t, x(t), \ x(t+s)) \leqslant \mu_p(t, v(t, x), v_t), \\ v_q^*(t, \ x(t), \ x(t+s)) \geqslant \mu_q(t, v(t\, x), v_t) \geqslant 0. \end{cases}$$

Let $x(t_0, \sigma)(t)$ *be any solution of* (2.1) *existing to the right of* t_0 *such that*

$$(4.5)\qquad \begin{cases} v_p(t_0+s, \sigma(s)) \leqslant \psi_p(s), \\ v_q(t_0+s, \sigma(s)) \geqslant \psi_q(s), \end{cases} \qquad (-\tau \leqslant s \leqslant 0).$$

Then

$$(4.6)\qquad v_p(t, x(t_0, \sigma)(t)) \leqslant u_p(t_0, \psi)(t), \qquad v_q(t, x(t_0, \sigma)(t)) \geqslant u_q(t_0, \psi)(t)$$

for $(t, x(t)) \in Z$ *where* $u(t_0, \psi)(t)$ *is the* k max $(n-k)$ min *solution of the system* (3.1) *existing to the right of* t_0.

The proof of this lemma can be formulated by following the argument of the Theorem 1 of [2] and the Theorem 6 of [6]. The details are omitted. However, it is important to note that the conditions (i) to (iv) of (4.1) ensure that $(t, x(t_0, \sigma)(t)) \in Z$ for $t \geqslant t_0$. For, if we choose $t_0 \in J$ and $\sigma \neq \theta$ then we have from (4.4)

$$(4.7)\qquad v_q(t, x(t_0, \sigma)(t)) \geqslant v_q(t_0, \sigma(t_0+s)) > 0,$$

since v_i satisfies Lipschitz's condition. Further, since $v_q(t, 0) = 0$ for all $(t, x) \in \bar{Z} - Z$, it follows from (4.7) that $(t, x(t_0, \sigma)(t)) \in Z$ for $t \geqslant t_0$.

Remark 1. We note that when $k = n$ and $k = 0$ the k max $(n-k)$ min solution reduces to the maximal and minimal solutions of the system (3.1) respectively.

5. Stability Definitions

We concentrate below on the following stability properties. In order to avoid repetition, only a few definitions given in [7] are restated. The null solution of the system (2.2) is said to satisfy the following properties if:

(p_1) for any $0 < \varepsilon < H$, $t_0 \in J$, there exists a function $\delta = \delta(t_0, \varepsilon)$ which is continuous in t_0 for each ε such that $\| \sigma \|_c \leqslant \delta$ implies that $\| x(t_0, \sigma)(t) \|_p < \varepsilon$ for $t \geqslant t_0$ and there exists a $t_1 > t_0$ such that $\| x(t_0, \sigma)(t_1) \|_q \geqslant \varepsilon$;

(p_2) δ in (p_1) is independent of t_0;

(p_3) for any $0 < \varepsilon < H$, $\alpha \geqslant 0$ and $t_0 \in J$, there exist positive numbers $T = T(t_0, \varepsilon, \alpha)$ and $t_1 > t_0$ such that $\| x(t_0, \sigma)(t) \|_p < \varepsilon$ for $t - t_0 \geqslant T$ and $\| x(t_0, \sigma)(t) \|_q \geqslant \varepsilon$ whenever $\| \sigma \|_c \leqslant \alpha$;

(p_4) T in (p_3) is independent of t_0;

(p_5) for any $0 < \varepsilon < H$, $\alpha \geqslant 0$ and $t_0 \in J$, there exist a positive function $\delta = \delta(t_0, \varepsilon)$ which is continuous in t_0 for each ε and a positive number $T = T(t_0, \varepsilon, \alpha)$ such that $\| x(t_0, \sigma)(t) \|_p < \varepsilon$ for $t \geqslant t_0$ and $\| x(t_0, \sigma)(t) \|_q < \varepsilon$ for $t - t_0 \geqslant T$ whenever $\| \sigma \|_c \leqslant \min(\alpha, \delta)$;

(p_6) the δ and T in (p_5) are independent of t_0;

(p_7) in addition to the property (p_1), for a fixed $t_0 \geqslant 0$, there exist two functions $\lambda(\alpha) \in M$ and $\rho(\alpha, t_0) \in M$ such that

$$\int_{t_0}^{t} \lambda(\| x(t_0, \sigma)(\tilde{t}) \|_p) d\tilde{t} < \rho(t_0, \| \sigma \|_c)$$

for $t \geqslant t_0$.

Remark 2. The properties (p_1) and (p_2) include as special cases the definitions of x-stability and uniform x-stability respectively stated in [7]. The property (p_3) is partial quasi-asymptotic stability whereas (p_5) is a new type of stability and it combines the partial x-stability with partial quasi-asymptotic stability. (p_7) is a partial k-stability which eventually includes the definition of partial L^p-stability [4].

Note that all the above stability definitions are with respect to the k and $(n-k)$ components where $0 \leqslant k \leqslant n$ and hence include, as special cases, the existing stability properties.

Now, let $u(t_0, \psi)(t)$ be any solution of (3.1) existing to the right of t_0. We can similarly formulate the properties (p_1^*) to (p_7^*) with respect to this solution. For example, (p_1^*) will be

(p_1^*) given $0 < \varepsilon < H$ and $t_0 \in J$, there exists a positive function $d = d(t_0, \varepsilon)$, which is continuous in t_0 for each ε such that $\| u(t_0, \psi)(t) \|_p < \varepsilon$ for $t \geqslant t_0$ whenever $\| \psi \|_c \leqslant d$ and for $t > t_0$, $\| u(t_0, \psi)(t) \|_q$ is either unbounded or indeterminate.

6. Main Results

We prove the following theorems.

Theorem 1. *Let the assumptions of the lemma hold and the vector function v defined above be such that*

$$a(\| x \|_p) \leqslant \| v(t, x) \|_p, \qquad (t, x) \in Z,$$

where the function $a \in M$. Then the properties (p_1^*) *and* (p_3^*) *imply the properties* (p_1) *and* (p_3) *respectively.*

Proof. Following a similar argument as in [5] and introducing suitable changes there in, it is easy to prove that partial equi-stability and quasi-asymptotic stability of the solution $u(t_0, \psi)(t)$ imply the corresponding properties of the solution $x(t_0, \sigma)(t)$. We omit these details. Further, because of (4.6), we have

$$\| (v(t, x(t_0, \sigma))(t) \|_q \geqslant \| u(t_0, \psi)(t) \|_q, \qquad (t, x) \in Z. \tag{6.1}$$

By hypotheses in (p_1^*), $\| u(t_0, \psi)(t) \|_q$ is unbounded and $v_i(t, x)$ is bounded on Z by assumption (4.1). Hence the inequality (6.1) cannot hold if we assume that the null solution of the system (2.1) is stable, or quasi-asymptotically stable. Thus, we conclude that the null solution is unstable with respect to q-components. This completes the proof of the theorem.

Theorem 2. *Let the assumptions of the lemma hold and the vector function v defined above be such that*

$$a(\| x \|_p) \leqslant \| v(t, x) \|_p \leqslant b(\| x \|_p) \qquad (t, x) \in Z \tag{6.2}$$

where the functions $a \in M$ and $b \in M$. Then (p_2^*) and (p_4^*) imply the properties (p_2) and (p_4) respectively.

The proof of the theorem will be similar to that of Theorem 1 except that the assumption that $b \in M$ satisfying (6.2) will help to determine δ in (p_1) and T in (p_4) respectively independent of t_0.

We state the following two theorems without proof and employ the lemma proved in [5] for $N = n$.

Theorem 3. *Let the assumptions of the lemma in [5] hold and the vector function v defined above be such that*

$$a(\| x \|_p) \leqslant \| v(t, x) \|_p, \qquad \tilde{a}(\| x \|_q) \leqslant \| v(t, x) \|_q, \qquad (t, x) \in Z,$$

where $\| v(t, x) \|_q \to 0$ as $\| x \| \to 0$ for each $t \in J$, the functions $a \in M$ and $\tilde{a} \in M$. Then (p_5^*) implies the corresponding property (p_5).

Theorem 4. *Let the assumptions of Theorem 3 hold and in addition assume that there exists two functions $b \in M$ and $\tilde{b} \in M$ such that*

$$\| v(t, x) \|_p \leqslant b(\| x \|_p), \qquad \| v(t, x) \|_q \leqslant \tilde{b}(\| x \|_q), \qquad (t, x) \in Z.$$

Then (p_6^*) implies the corresponding property (p_6).

Theorem 5. *Let the assumptions of the lemma hold and the vector function v described above is such that*

$$a(\| x \|_p) \leqslant \| v(t, x) \|_p \leqslant b(\| x \|) \tag{6.3}$$

where the functions $a \in M$ and $b \in M$. Then the property (p_7^*) implies (p_7).

Proof. Let $0 < \varepsilon < H$ and $t_0 \in J$. Since (p_1^*) holds, given $a(\varepsilon) > 0$, we have, a positive function $d = d(t_0, \varepsilon)$ such that $\| u(t_0, \psi)(t) \|_p < a(\varepsilon)$ for $t \geqslant t_0$ whenever

$$\| \psi \|_c \leqslant d. \tag{6.4}$$

Since $\| v(t, x) \|_p \to 0$ as $\| x \| \to 0$ for each t, we have, in view of (4.6) and (6.4), a $\delta = \delta(t_0, \varepsilon)$ such that $\| v(t_0 + s, \sigma(s)) \|_p \leqslant d$ implies $\| \sigma \|_c \leqslant \delta$. With this δ the property (p_1) holds. Now, fix $t_0 \in J$. Because of (p_7^*), we have, two functions $\tilde{\lambda} \in M$ and $\tilde{\rho} \in M$ and $d > 0$ such that

$$\int_{t_0}^{t} \tilde{\lambda}(\| u(t_0, \psi)(z) \|_p) \mathrm{d}z \leqslant \tilde{\rho}(t_0, \| \psi \|_c) \tag{6.5}$$

whenever $0 \leqslant \| \psi \|_c \leqslant d$. Now, choose δ as above and let $\max\limits_{-\tau \leqslant s \leqslant 0} \| v_{t_0} \|_p = \| \psi \|_c$. Then, we have

$$\int_{t_0}^{t} \tilde{\lambda}(a(\| x(t_0, \sigma)(z) \|_p)) \mathrm{d}z \leqslant \int_{t_0}^{t} \tilde{\lambda}(\| v(z, x(t_0, \sigma)(z)) \|_p) \mathrm{d}z \leqslant$$

$$\leqslant \int_{t_0}^{t} \tilde{\lambda}(\| u(t_0, \psi)(z) \|_p) \mathrm{d}z \leqslant \tilde{\rho}(t_0, \| \psi \|_c) =$$

$$= \tilde{\rho}(t_0, \max_{-\tau \leqslant s \leqslant 0} \| v_{t_0} \|_p) \leqslant \tilde{\rho}(t_0, b(\| \sigma \|_c))$$

because of (4.6), (6.3) and (6.5). Now, choose $\lambda = \tilde{\lambda} \circ a$ and $\rho = \tilde{\rho} \circ b$. With this choice, we conclude that (p_7) holds.

Remark 3. In the section **5**, we have mentioned only a few definitions of stability. This list can be elongated by additing a number of other types of stabilities already studied. Similarly these results can be extended to perturbed systems by introducing suitable changes in the given theorems.

Remark 4. Lakshmikantham [5], Pieffer and Rouche [7] have given the definitions of conditional stability and partial stability respectively. It will be observed that, in the first case, the initial condition is restricted to a certain subset of R^n whereas in the second case some components of the solution have a desired behaviour. Taking into account the role of (t, x) and (t_0, x_0), it appears that the concepts of conditional stability and partial stability are corelated. This relationship is not yet known.

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STABILITATEA SISTEMELOR DIFERENŢIALE EREDITARE

(Rezumat)

Se extind rezultatele lui Pieffer şi Rouche [7] care au utilizat metoda funcţiei Liapunov pentru studiul comportării parţial stabile a sistemelor diferenţiale ordinare. În scopul studierii sistemelor diferenţiale ereditare, autorul introduce noi concepte de stabilitate, propune definiţii pentru stabilităţile parţiale şi utilizează metoda a doua a lui Liapunov asociată cu o inegalitate diferenţială (care conţine drept cazuri particulare inegalităţile lui Wazewsky [8] şi Lakshmikantham şi Shendge [6]).

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SUR L'EXISTENCE ET LA STABILITÉ DE LA SOLUTION DE L'ÉQUATION INTÉGRALE NON-LINÉAIRE DE VOLTERRA

PAR

ADRIAN CORDUNEANU

1. Le but du travail est d'établir quelques résultats sur la solution de certaines équations sur le demiaxe $t \geqslant 0$. Nous avons besoin des lemmes suivants :

Lemme 1 [1]. *X et Y étant deux espaces de Banach, u une application linéaire de X sur Y satisfaisant à la condition $\| x \| \leqslant m \| u(x) \|$, $\forall x \in X$, p une application (non-linéaire) de X dans Y satisfaisant à la condition $\| p(x_1) - p(x_2) \| \leqslant L \| x_1 - x_2 \|$, $\forall x_1, x_2 \in X$, il existe une seule solution x de l'équation $u(x) + p(x) = y$, y étant un élément fixé de Y, dès que $mL < 1$.*

Démonstration. La condition imposée sur u assure l'existence de u^{-1} et l'inégalité $\| u^{-1} \| \leqslant m$. Considérons l'opérateur $T : X \to X$ défini par la relation $Tx = u^{-1}(y) - u^{-1}(p(x))$. Il est aisé de voir que T est contractant (la constante de contraction étant $mL < 1$) et donc il existe un seul point $x \in X$ tel que $x = Tx$, ce qui équivaut à $u(x) + p(x) = y$.

Lemme 2 [1]. *X, Y et u étant comme dans le lemme précédent, p étant défini sur $S = \{ x \, ; \, x \in X, \| x \| \leqslant \gamma \}$ à valeurs dans Y, $\| p(x_1) - p(x_2) \| \leqslant L \| x_1 - x_2 \|$, $\forall x_1, x_2 \in S$, $m(L\gamma + \| p(0) \|) < \gamma$, il existe une seule solution $x \in S$ de l'équation $u(x) + p(x) = y$, $y \in Y$ satisfaisant à la condition $\| y \| < \gamma m^{-1} - (L\gamma + \| p(0) \|)$.*

Lemme 3 [2]. *X, Y, S et u étant comme dans le lemme 2, p une application de S dans Y avec $\overline{p(S)}$ compacte dans Y, $d = \sup \| p(x) \|$ pour $x \in S$, $\gamma - md > 0$, il existe au moins une solution $x \in S$ de l'équation $u(x) + p(x) = y$, $y \in Y$ satisfaisant à la majoration $\| y \| < \gamma m^{-1} - d$.*

Démonstration. On applique le principe du point fixe de Schauder à l'opérateur $Tx = u^{-1}(y) - u^{-1}(p(x))$ qui satisfait à la condition $TS \subseteq \overline{TS} \subseteq S$, $\overline{TS}$ étant un ensemble compact dans X.

D é f i n i t i o n. Désignons par $C_n[0, +\infty)$ l'espace de Banach des fonctions-vecteurs (à n composantes) continues et bornées pour $t \geqslant 0$, où la norme d'un élément x est donnée par $\| x \| = \sup_{t \geqslant 0} \| x(t) \|$. La norme d'un vecteur est celle euclidienne.

L e m m e 4 [3]. *La condition nécessaire et suffisante pour que l'ensemble $\mathcal{M} \subseteq C_n[0, +\infty)$ soit relativement compact est que les fonctions $x \in \mathcal{M}$ soient uniformément bornées : $\| x \| \leqslant M$, $\forall x \in \mathcal{M}$ et que pour tout $\varepsilon > 0$ il existe un recouvrement fini de $[0, +\infty)$ par ensembles ouverts, tels que l'oscillation de toute fonction $x \in \mathcal{M}$ soit inférieure à ε dans chaque ensemble du recouvrement.*

Dans ce qui suit nous emploierons aussi les notations suivantes : Δ pour l'ensemble $\{(t, s) ; 0 \leqslant s \leqslant t < +\infty\}$; Δ' pour l'ensemble $\{(t, s) ; 0 \leqslant t \leqslant s < +\infty\}$; R pour l'ensemble des nombres réels ; R^n pour $R \times \times R \times \ldots \times R$; I pour l'opérateur identité dans un espace de Banach.

2. P r o p o s i t i o n 1. *Considérons l'équation*

$$x(t) = y(t) + \int_0^t [k(t, s)\, x(s) + p(t, s, x(s))]\, \mathrm{d}s, \quad t \geqslant 0 \tag{1}$$

et admettons les hypothèses suivantes :

(i) *$k(t, s)$ est une fonction-matrice du type $(n \times n)$, continue dans Δ,*

$$\sup_{t \geqslant 0} \int_0^t \| k(t, s) \| \, \mathrm{d}s < +\infty, \qquad \sup_{t \geqslant 0} \int_0^t \| r(t, s) \| \, \mathrm{d}s = \Gamma < +\infty, \tag{2}$$

$r(t, s)$ étant le noyau résolvant de l'équation

$$x(t) = y(t) + \int_0^t k(t, s)\, x(s)\, \mathrm{d}s, \quad t \geqslant 0 ; \tag{3}$$

(ii) *$p(t, s, x)$ est une fonction-vecteur du type $(n \times 1)$ continue sur $\Delta \times R^n$, $p(t, s, 0)$ satisfait à la majoration (2) et de plus*

$$\| p(t, s, x_1) - p(t, s, x_2) \| \leqslant q(t, s) \| x_1 - x_2 \|, \quad (t, s) \in \Delta, \quad y_1, y_2 \in R^n, \tag{4}$$

$$\sup_{t \geqslant 0} \int_0^t q(t, s)\, \mathrm{d}s = Q < +\infty, \quad Q(1 + \Gamma) < 1. \tag{5}$$

Il résulte alors que l'équation (1) admet une solution unique $x \in \in C_n[0, +\infty)$, quel que soit $y \in C_n[0, +\infty)$ fixé.

Démonstration. En forme abstraite, l'équation (1) s'écrit $x = y + Kx + Px$, K étant l'opérateur linéaire intégral ayant le noyau $k(t, s)$ et P étant l'opérateur défini par la relation

$$Px(t) = \int_0^t p(t, s, x(s))\, ds, \quad t \geqslant 0. \tag{6}$$

Grâce au conditions imposées, K et P sont définis sur $C_n[0, +\infty)$ à valeurs dans lui-même. L'opérateur $I-K$ joue le rôle de u (lemme 1), m étant égal à $1 + \Gamma$. Le rôle de p est tenu ici par P. Compte tenu aussi des dernières conditions, on arrive à la conclusion que le lemme 1 est applicable. La solution x est la limite uniforme sur $t \geqslant 0$ des approximations successives ; on peut prendre pour point initial n'importe quelle fonction-vecteur continue et bornée pour $t \geqslant 0$.

Remarque. La norme d'une matrice est celle euclidienne.

Proposition 2. *L'équation*

$$x'(t) = A(t)x(t) + p(t, x(t)), \quad x(0) = x_0, \tag{7}$$

admet solution unique pour $t \geqslant 0$ (vecteur du type $(n \times 1)$), satisfaisant à la majoration $\| x(t) \| < \gamma$ si les conditions suivantes sont remplies :

(i) *la fonction-matrice $A(t)$ du type $(n \times n)$ est continue pour $t \geqslant 0$ et de plus* $\exp\left(\int_0^\infty \| A(t) \| \, dt\right) < +\infty$;

(ii) *la fonction-vecteur $p(t, x)$ du type $(n \times 1)$ est continue pour $t \geqslant 0$,* $\| x \| \leqslant \gamma$,

$$\| p(t, x_1) - p(t, x_2) \| \leqslant q(t) \| x_1 - x_2 \|, \; t \geqslant 0, \; \| x_i \| \leqslant \gamma, \; (i = 1, 2), \tag{8}$$

$$\int_0^\infty \| p(t, 0) \| \, dt = \bar{p} < +\infty, \quad \int_0^\infty q(t)\, dt = Q < +\infty, \quad Q \exp\left(\int_0^\infty \| A(t) \| \, dt\right) < 1 ; \tag{9}$$

$$\text{(iii)} \qquad (Q\gamma + \bar{p}) \exp\left(\int_0^\infty \| A(t) \| dt\right) < \gamma,$$

$$\| x_0 \| < \gamma \exp\left(-\int_0^\infty \| A(t) \| \, dt\right) - (Q\gamma + \bar{p}).$$

D é m o n s t r a t i o n. L'équation (7) est équivalente à

$$x(t) = x_0 + \int_0^t [A(s)\,x(s) + p(s, x(s))]\,\mathrm{d}s, \quad t \geqslant 0. \tag{10}$$

Avec les notations introduites dans l'énoncé et dans la démonstration de la proposition précédente on obtient ($k(t, s) = A(s)$)

$$\sup_{t \geqslant 0} \int_0^t \| r(t, s) \|\,\mathrm{d}s = \exp\left(\int_0^\infty \| A(t) \|\,\mathrm{d}t \right) - 1. \tag{11}$$

En posant $u = I - K$, $p = P$, $X = Y = C_n[0, +\infty)$, S étant la boule fermée de rayon γ et centre O dans $C_n[0, +\infty)$, on voit que toutes les conditions du lemme 2 sont remplies et donc l'équation (10) admet une solution unique $x \in S$.

P r o p o s i t i o n 3. *Considérons l'équation*

$$x(t) = y(t) + \int_t^\infty [k(t, s)x(s) + p(t, s, x(s))]\,\mathrm{d}s, \quad t \geqslant 0, \tag{12}$$

et supposons que les hypothèses suivantes sont satisfaites:

(i) $k(t, s)$ *est une fonction-matrice du type* $(n \times n)$ *continue sur* Δ', *satisfaisant à la majoration* $\| k(t, s) \| \leqslant a(t)\,b(s)$, a *et* b *étant deux fonctions continues non-négatives sur* $[0, +\infty)$,

$$\sup_{t \geqslant 0} a(t) \int_t^\infty b(s) \exp\left(\int_t^s a(\tau)\,b(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s = B < +\infty; \tag{13}$$

(ii) $p(t, s, x)$ *est une fonction-vecteur du type* $(n \times 1)$ *continue sur* $\Delta' \times R^n$,

$$\begin{gathered} \sup_{t \geqslant 0} \int_t^\infty \| p(t, s, 0) \|\,\mathrm{d}s < +\infty, \\ \| p(t, s, x_1) - p(t, s, x_2) \| \leqslant q(t, s) \| x_1 - x_2 \|, \quad (t, s) \in \Delta', \quad x_1, x_2 \in R^n, \end{gathered} \tag{14}$$

$$\sup_{t \geqslant 0} \int_t^\infty q(t, s)\,\mathrm{d}s = Q < +\infty, \quad Q(1 + B) < 1. \tag{15}$$

Il résulte alors que l'équation (12) *admet une solution unique* $x \in C_n[0, +\infty)$, *quel que soit* $y \in C_n[0, +\infty)$ *fixé.*

Proposition 4. *Considérons l'équation*

$$x(t) = y(t) + \int_0^\infty [k(t, s)x(s) + p(t, s, x(s))]\, ds, \quad t \geqslant 0 \tag{16}$$

et admettons que les conditions suivantes sont remplies :

(i) *la condition* (i) *de la proposition* 1 *est vérifiée et de plus*

$$\sup_{t \geqslant 0} \int_t^\infty \| k(t, s) \| \, ds = k_1 < +\infty, \quad k_1(1 + \Gamma) < 1; \tag{17}$$

(ii) *la fonction-vecteur* $p(t, s, x)$ *du type* $(n \times 1)$ *est continue sur* $\Delta' \times R^n$,

$$\sup_{t \geqslant 0} \int_0^\infty \| p(t, s, 0) \| \, ds < +\infty, \tag{18}$$

$$\| p(t, s, x_1) - p(t, s, x_2) \| \leqslant q(t, s) \| x_1 - x_2 \|, \quad (t, s) \in \Delta', \quad x_1, x_2 \in R^n$$

$$\sup_{t \geqslant 0} \int_0^\infty q(t, s)\, ds = Q < +\infty, \quad Q + k_1 < 1 + \Gamma. \tag{19}$$

Il résulte alors que l'équation (16) *admet une solution unique* $x \in C_n[0, +\infty)$, *quel que soit* $y \in C_n[0, +\infty)$ *fixé.*

Démonstration. Désignons par K, K_1, K_2 les opérateurs intégraux donnés par les relations

$$Kx(t) = \int_0^\infty k(t, s)x(s)\, ds, \quad K_1 x(t) = \int_0^t k(t, s)\, x(s)\, ds, \tag{20}$$

$$K_2\, x(t) = \int_t^\infty k(t, x)\, x(s)\, ds, \quad t \geqslant 0,$$

qui sont tous définis et bornés dans l'espace $C_n[0, +\infty)$. On déduit que $u = I - K$ est surjectif, $u^{-1} = ((I - K_1) - K_2)^{-1}$ existe et $m = (1 + \Gamma) \cdot (1 - k_1(1 + \Gamma))^{-1}$. On applique le lemme 1 pour achever la démonstration.

Proposition 5. *Considérons l'équation* (1) *et supposons que*

(i) *la condition* (i) *de la proposition 1 est remplie ;*

(ii) $p(t, s, x)$ *est une fonction-vecteur du type* $(n \times 1)$ *continue pour* $(t, s) \in \Delta$, $\| x \| \leqslant \gamma$ *et de plus*

$$(21)\quad \sup_{\|x\|\leqslant\gamma} \| p(t, s, x) \| \leqslant \alpha(t)\, \beta(s), \quad \sup_{t\geqslant 0} \alpha(t) \int_0^t \beta(s) \mathrm{d}s = d, \quad (1 + \Gamma)d < \gamma,$$

$$(22)\qquad \lim_{t\to\infty} \alpha(t) \int_0^t \beta(s)\, \mathrm{d}s = 0.$$

Il résulte alors que l'équation (1) *admet au moins une solution* $x \in C_n[0, +\infty)$, $\| x \| \leqslant \gamma$ *si* $y \in C_n[0, +\infty)$, $\| y \| < \gamma(1 + \Gamma)^{-1} - d$.

D é m o n s t r a t i o n. Nous appliquerons le lemme 3. L'équation (1) s'écrit $x = y + Kx + Px$. En posant $u = I - K$, on trouve $m = 1 + \Gamma$. D'autre part, $\sup \| Px \| \leqslant d$ pour $x \in S$, S étant la boule fermée de rayon γ et centre O dans cet espace. Il reste seulement de montrer que $\overline{P(S)}$ est compacte dans $C_n[0, +\infty)$. En vérité la famille des fonctions Px, $x \in S$ est uniformément bornée pour $t \geqslant 0$ et pour tout $\varepsilon > 0$ on a :

$$\| Px(t_2) - Px(t_1) \| \leqslant \int_{t_1}^{t_2} \| p(t_2, s, x(s)) \| \mathrm{d}s +$$

$$(23)\qquad + \int_0^{t_1} \| p(t_2, s, x(s)) - p(t_1, s, x(s)) \| \mathrm{d}s \leqslant \alpha(t_2) \int_0^{t_2} \beta(s)\, \mathrm{d}s +$$

$$+ \alpha(t_2) \int_0^{t_2} \beta(s) \mathrm{d}s + \alpha(t_1) \int_0^{t_1} \beta(s)\, \mathrm{d}s < \varepsilon, \quad t_2 \geqslant t_1 > A(\varepsilon).$$

Compte tenu de la continuité uniforme de $p(t, s, x)$ sur l'ensemble $0 \leqslant s \leqslant t \leqslant A(\varepsilon) + 1$, $\| x \| \leqslant \gamma$ on obtient que l'intervalle $[0, A(\varepsilon) + 1]$ peut être couvert avec un nombre fini d'intervalles dans lesquels l'oscillation de chaque fonction Px, $x \in S$ est inférieure à ε. Il résulte $\overline{PS}$ compacte d'après le lemme 4.

Remarque. Si $p(t, s, x)$ ne dépend pas de t, les relations (21) seront remplacées par

$$(21')\quad \sup_{\|x\|\leqslant\gamma} \| p(s, x) \| \leqslant \beta(s), \quad \int_0^\infty \beta(s)\, \mathrm{d}s = d < +\infty, \quad (1 + \Gamma)d < \gamma.$$

3. P r o p o s i t i o n 6. *Considérons l'équation scalaire*

$$(24)\qquad x(t) = y(t) + \int_0^t k(t, s, x(s))\, \mathrm{d}s, \quad t \geqslant 0$$

et supposons que

(i) *la fonction* $k(t, s, x)$ *est continue pour* $(t, s) \in \Delta$, $|x| \leqslant \gamma$, *monotone croissante par rapport à* x, $k(t, s, 0) = 0$;

$$\text{(ii)} \quad \sup_{\|x\| \leqslant \gamma} |k(t, s, x)| \leqslant \alpha(t)\,\beta(s), \quad \sup_{t \geqslant 0} \alpha(t) \int_0^t \beta(s)\mathrm{d}s = d < \gamma,$$

$$\lim_{t \to \infty} \alpha(t) \int_0^t \beta(s)\,\mathrm{d}s = 0 ;$$

(iii) *la solution* x *de l'équation* (24) *si elle existe, est unique, quel que soit* $y \in C_1[0, +\infty)$.

Il résulte alors que l'équation (24) *admet une seule solution* $x \in C_1[0, +\infty)$, *quel que soit* $y \in C_1[0, +\infty)$, $\|y\| < \gamma - d$. *La solution* x *est stable par rapport à* y.

D é m o n s t r a t i o n . Considérons l'opérateur T_c, donné par la relation :

$$(25) \qquad T_c\,x(t) = c + \int_0^t k(t, s, x(s))\mathrm{d}s, \quad t \geqslant 0, \quad c = \text{const}, \quad |c| < \gamma - d,$$

défini sur S, la boule fermée de rayon γ et centre O dans $C_1[0, +\infty)$. Il est aisé de voir que $T_cS \subseteq \overline{T_cS} \subseteq S$ et que $\overline{T_cS}$ est compacte dans $C_1[0, +\infty)$. Le théorème de S c h a u d e r assure l'existence au moins d'un point fixe $x \in S$ de l'opérateur T_c. En outre, la correspondance $c \rightleftarrows x$ est biunivoque et l'application $x \to c$ est continue. Mais l'ensemble $\mathscr{X} = \{x ; x \in C_1[0, +\infty), x = T_cx, |c| < \gamma - d\}$ est relativement compact dans $C_1[0, +\infty)$, c'est-à-dire $\overline{\mathscr{X}}$ est compact, et donc l'application $c \to x$ est aussi continue. On a donc $\|x\| < \varepsilon$ si $|c| < \delta(\varepsilon)$. D'autre part, si y est une fonction appartenant à $C_1[0, +\infty)$, $\|y\| < \gamma - d$, l'équation (24) admet une solution unique pour $t \geqslant 0$, satisfaisant à la majoration $\|x\| < \gamma$. (On montre que la solution x qui existe sur tout intervalle fini, peut être prolongée sur le demiaxe $t \geqslant 0$). Considérons les relations

$$(26) \qquad \begin{aligned} v(t) &= \|y\| + \int^t k(t, s, v(s))\,\mathrm{d}s, \\ w(t) &= -\|y\| + \int_0^t k(t, s, w(s))\,\mathrm{d}s, \quad t \geqslant 0. \end{aligned}$$

Compte tenu de la monotonie de $k(t, s, x)$ par rapport à x, on obtient $w(t) \leqslant x(t) \leqslant v(t)$ pour $t \geqslant 0$, d'où résulte $\|x\| < \varepsilon$ si $\|y\| < \delta(\varepsilon)$, x étant la solution de l'équation (24).

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DESPRE EXISTENŢA ŞI STABILITATEA SOLUŢIEI ECUAŢIEI INTEGRALE NELINIARE VOLTERRA

(Rezumat)

Se aplică lemele 1, 2 şi 3 pentru a deduce existenţa soluţiei unor ecuaţii integrale neliniare pe interval infinit ($t \geq 0$). Se foloseşte de asemenea un criteriu de compactitate în $C[0, \infty)$, dat de lema 4. Ultima propoziţie se referă la stabilitatea soluţiei ecuaţiei (24).

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EIN KRITERIUM FÜR DIE P-SEPARIERBARKEIT DER LAPLACE-GLEICHUNG IN RIEMANNSCHEN RÄUMEN

VON

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1. Einleitung

Es gibt partielle Differentialgleichungen 2. Ordnung, die vermöge eines Produktansatzes der Lösungsfunktion U nicht direkt, jedoch nach einer konformen Transformation derselben: $V(x) = P(x).U(x)$, $P(x) \neq 0$, separierbar sind. Derartige Differentialgleichungen heissen P-separierbar. Die Klasse der Separierbarkeit gestattenden partiellen Differentialgleichungen 2. Ordnung wird auf diese Weise wesentlich erweitert. Für die P-Separierbarkeit der Laplace-Gleichung hat H. Ober-Kassebaum [1] gewisse Kriterien angegeben. Sie sind jedoch für die praktische Entscheidung, ob P-Separierbarkeit vorliegt oder nicht, wenig geeignet. Dagegen hat Steigenberger [2] Kriterien für die vollständige Separierbarkeit der Laplace-Gleichung hergeleitet, die diese Entscheidung gestatten. In der vorliegenden Arbeit werden diese Kriterien auf die P-Separierbarkeit der Laplace-Gleichung erweitert. Bezüglich Definitionen und Vereinbarungen wird auf [2] verwiesen.

2. P-Separation der Laplace-Gleichung

Definition. Die Laplace-Gleichung

$$\Delta U \equiv g^{ij} \nabla_i \nabla_j U \equiv \frac{1}{\sqrt{|g|}} [\sqrt{|g|}\, g^{ij} U_{,i}]_{,j} = 0 \tag{1}$$

heisst in dem festen Koordinatensystem $(x^1, \ldots, x^n)$ des V^n P-separierbar' wenn es eine Funktion $P = P(x) \neq 0$, gibt, so dass die vermöge der konformen Transformation $V = P.U$ aus (1) entstehende Gleichung produkt-

separierbar ist, wobei $n-1$ Separationskonstanten a_i in die durch Separation gewonnenen gewöhnlichen Differentialgleichungen 2. Ordnung

$$X_i(U_i) \equiv \frac{d^2U_i}{(dx^i)^2} + F_i\left(a_1, \ldots, a_{n-1};\ x^i;\ U_i;\ \frac{dU_i}{dx^i}\right) = 0$$

eingehen und

$$\text{Rang} \left\| \frac{\partial F_i}{\partial a_j} \right\| = n-1, \quad (i = 1, \ldots, n;\ j = 1, \ldots, n-1)$$

ist.

Ober-Kassebaum [1] bewies 1964 u. a. den folgenden

Satz 1. *Die Laplace-Gleichung* (1) *ist in dem Koordinatensystem* $(x^1, \ldots, x^n)$ *des* V^n *genau dann P-separierbar, wenn es Funktionen* $P(x) \neq 0$, $Q(x) > 0$, $q_\lambda(x^\lambda)$, $f_\lambda(x^\lambda) \neq 0$, $\psi^\varkappa_\lambda(x^\lambda)$ *mit* $\Psi \equiv \det(\psi^\varkappa_\lambda) \neq 0$ *gibt, so dass gilt:*

(A) $$g^{\varkappa\lambda} = g^\varkappa \delta^\varkappa_\lambda,$$

(B) $$\psi^\mu_i g^i = \delta^\mu_1 Q,$$

(C) $$\frac{1}{P} \sum_{\varkappa=1}^{n} \frac{g^\varkappa}{f_\varkappa} (f_\varkappa P_{,\varkappa})_{,\varkappa} = g^i q_i,$$

(D) $$\frac{P^2 \Psi}{\sqrt{|g|}\, Q} = \prod_{i=1}^{n} \frac{1}{f_i}.$$

(A) drückt die Orthogonalität der Metrik aus. Metriken, die der Bedingung (B) genügen, heissen konform-Stäckelsch, für $Q = 1$ Stäckelsch. (C) und (D) stellen Beschränkungen für die Funktionen P und Q dar. Wegen der unbekannten Funktionen $\psi^\varkappa_\lambda$, q_λ, f_λ, P und Q ist Satz 1 nicht geeignet zu entscheiden, ob eine gegebene Laplace-Gleichung P-separierbar ist oder nicht. Zur Konstruktion von P-separierbaren Gleichungen kann Satz 1 jedoch gut verwendet werden. Die oben genannten Funktionen sind dabei so vorzugeben, dass die Bedingungen (A)...(D) erfüllt sind.

Im folgenden wird gezeigt, wie die unbekannten Funktionen aus den Bedingungen (B), (C) und (D) eliminiert werden können. Hierbei werden die Lemmata 1, 2 und 3 aus [2] wesentlich benutzt.

Da die Metrik

(2) $$G^\mu := Q^{-1} g^\mu$$

Stäckelsch ist, gilt nach Lemma 1, [2], $\bar{L}_{\varkappa\lambda}[G^\mu] = 0$, $\varkappa \neq \lambda$, wobei $\bar{L}_{\varkappa\lambda}$ der in [2] definierte Differentialoperator bzgl. der Metrik G^μ ist.

Der dieser Stäckelmetrik zugeordnete Ricci-Tensor [1]) hat die Komponenten

[1]) Siehe [1, S. 42].

$$\bar{R}_{\varkappa\lambda} = \frac{3}{2}\left[\log\left|\frac{\sqrt{|G|}}{\Psi}\right|\right]_{,\lambda\varkappa}, \quad (\varkappa \neq \lambda), \tag{3}$$

mit $G \equiv \det(G_i)$. Aus (D) folgt nach Logarithmieren und zweimaliger partieller Differentiation

$$\bar{R}_{\varkappa\lambda} = \frac{3}{2}[\log(P^2 Q^{\frac{n-2}{2}})]_{,\varkappa\lambda}, \quad (\varkappa \neq \lambda). \tag{D'}$$

Nach Ober-Kassebaum lässt sich damit Satz 1 wie folgt formulieren:

S a t z 1'. *Notwendig und hinreichend für die P-Separierbarkeit der Gleichung* (1) *in dem Koordinatensystem* $(x^1, \ldots, x^n)$ *des* V^n *ist die Existenz von* $2n + 2$ *Funktionen* $q_\lambda(x^\lambda)$, $f_\lambda(x^\lambda) \neq 0$, $P(x) \neq 0$ *und* $Q(x) > 0$ *derart, dass gilt:*

$$g^{\varkappa\lambda} = g^\varkappa \delta^\varkappa_\lambda, \tag{A}$$

$$\bar{L}_{\varkappa\lambda}[G^\mu] = 0, \quad (\varkappa \neq \lambda), \tag{B'}$$

$$\frac{1}{P}\sum_{\varkappa=1}^{n}\frac{g_\varkappa}{f_\varkappa}(f_\varkappa P_{,\varkappa})_{,\varkappa} = g^i q_i, \tag{C}$$

$$\frac{3}{2}[\log(P^2 Q^{\frac{n-2}{2}})]_{,\varkappa\lambda} = \bar{R}_{\varkappa\lambda}, \quad (\varkappa \neq \lambda). \tag{D'}$$

Nach Lemma 2, [2], ist die Metrik g^μ konform-Stäckelsch genau dann, wenn gilt

$$g_\mu L_{\varkappa\lambda}[g^\mu] = f_{\varkappa\lambda}(x), \quad (\varkappa \neq \lambda). \tag{B''}$$

Die Funktionen $f_{\varkappa\lambda}$ hängen nicht von μ ab, und der Faktor Q genügt der Beziehung

$$Q^{-1} L_{\varkappa\lambda}[Q] = f_{\varkappa\lambda}(x). \tag{4}$$

(B'') ist die zu (B) bzw. (B') äquivalente und tatsächlich überprüfbare Bedingung.

Aus der Bedingung (D'), die ein partielles Differentialgleichungssystem zur Bestimmung der Funktion P darstellt kann die unbekannte Funktion Q eliminiert werden, indem der Ricci-Tensor $\bar{R}_{\varkappa\lambda}$ bzgl. der nicht bekannten Metrik G^μ ausgedrückt wird durch den Ricci-Tensor $R_{\varkappa\lambda}$ bzgl. der durch (1) gegebenen, nach (B'') notwendig konform-Stäckelschen Metrik g^μ.

Die konforme Transformation für die Metrik g^μ sei durch (2) definiert mit $Q \equiv \exp[-2\sigma(x)]$. Dann gilt für die Ricci-Tensoren

$$\bar{R}_{\varkappa\lambda} = R_{\varkappa\lambda} - (n-2)(\nabla_\varkappa \sigma_{,\lambda} + \sigma_{,\varkappa}\sigma_{,\lambda}), \quad (\varkappa \neq \lambda). \tag{5}$$

Der Ricci-Tensor $R_{\varkappa\lambda}$ hat in beliebigen orthogonalen Koordinaten bei Verwendung des in [2] definierten Differentialoperators $L_{\varkappa\lambda}$ die Gestalt

$$(6) \qquad R_{\varkappa\lambda} = \frac{1}{4} \sum_{\mu \neq \varkappa,\lambda}^{n} [g_\mu L_{\varkappa\lambda}[g^\mu] - 3 (\log | g^\mu |)_{,\varkappa\lambda}], \quad (\varkappa \neq \lambda).$$

Ebenfalls unter Verwendung des Differentialoperators $L_{\varkappa\lambda}$ und mit expliziter Darstellung der kovarianten Ableitung gilt

$$(7) \qquad \nabla_\varkappa \sigma_{,\lambda} + \sigma_{,\varkappa} \sigma_{,\lambda} \equiv -\frac{1}{2} \{L_{\varkappa\lambda} [\sigma] - 2\sigma_{,\varkappa} \sigma_{,\lambda} - 3 \sigma_{,\varkappa\lambda}\}.$$

Mit Hilfe der identischen Umformungen

$$[\log Q^{\frac{n-2}{2}}]_{,\varkappa\lambda} \equiv -(n-2)\sigma_{,\varkappa\lambda},$$

$$Q^{-1} L_{\varkappa\lambda}[Q] \equiv -2\{L_{\varkappa\lambda}[\sigma] - 2\sigma_{,\varkappa} \sigma_{,\lambda}\},$$

sowie den Beziehungen (B″), (4), (5), (6) und (7) kann die Gleichung (D′) in der Form

$$(\mathrm{D}'') \qquad (\log P^2)_{,\varkappa\lambda} = \frac{1}{2} \left(\log \prod_{\substack{i=1 \\ i \neq \varkappa,\lambda}}^{n} | g_i | \right)_{,\varkappa\lambda} \equiv \frac{1}{2} [\log | g\, g^\varkappa g^\lambda |]_{,\varkappa\lambda}, \quad (\varkappa \neq \lambda)$$

geschrieben werden. Die Bedingung (D'') gestattet, die Funktion $P(x)$ durch Integration zu gewinnen unter der Voraussetzung, dass gewisse Integrabilitätsbedingungen erfüllt sind.

Für die P-Separierbarkeit der Laplace-Gleichung (1) muss die Metrik g^μ nach (B″) notwendig konform-Stäckelsch sein.

Aus der Beziehung (B″) folgt für $\mu = \lambda$ und $\mu = \varkappa$

$$8) \qquad (\log | g^\lambda |)_{,\varkappa\lambda} = (\log | g^\varkappa |)_{,\varkappa\lambda}.$$

Vermöge (8) wird aus (D″)

$$(9) \qquad \left[\log \left(\frac{P^2}{\sqrt{|g|}\, |g^\varkappa|} \right) \right]_{,\varkappa\lambda} = 0, \quad (\varkappa \neq \lambda).$$

Also ist

$$\left[\log \left(\frac{P^2}{\sqrt{|g|}} \right) \right]_{,\varkappa} = h_\varkappa(x^\varkappa) + (\log | g^\varkappa |)_{,\varkappa},$$

wo $h_\varkappa(x^\varkappa)$ beliebige einargumentige stetige Funktionen sind.

Die Integrabilitätsbedingungen für das partielle Differentialgleichungssystem (9) sind wegen (8) erfüllt. Integration von (9) ergibt

$$P^2 = \sqrt{|g|}\exp\left[\int^{(x)}\sum_{\varkappa=1}^{n}(\log|g^{\varkappa}|)_{,\varkappa}\,dx^{\varkappa}\right]\prod_{i=1}^{n} t_i(x^i), \tag{10}$$

wobei das im Exponenten stehende Linienintegral wegunabhängig ist. Als notwendige Bedingung für die P-Separierbarkeit der Laplace-Gleichung ist (B″) also gleichzeitig hinreichend für die Integrierbarkeit des Systems (D″), dh. für die Bestimmung einer Funktion P gemäss (D).

Zu berechnen ist das im Exponenten stehende Linienintegral, die Funktionen $t_\varkappa \equiv \exp\left[\int h_\varkappa(x^\varkappa)dx^\varkappa\right]$ sind beliebig. Die Funktion P ist damit, falls g^μ (B″) erfüllt, durch die Bedingung (D″) bis auf einen Produktfaktor eindeutig bestimmt. (D″) ist nun keine Bedingung im obigen Sinne mehr; sie ist entweder mit (B″) erfüllt oder die Laplace-Gleichung ist nicht P-separierbar.

Aus der Bedingung (C) des Satzes 1 (bzw. Satz 1′) sind noch die unbekannten Funktionen $f_\varkappa$ und $q_\varkappa$ zu eliminieren. Angenommen, eine der Funktionen (10) genügt der Bedingung (C). Dann genügt $\bar{P} := P\prod_{\varkappa=1}^{n} p_\varkappa(x^\varkappa)$ mit $\bar{f}_\varkappa := f_\varkappa p_\varkappa^{-2}$ der Relation

$$\frac{1}{\bar{P}}\sum_{\varkappa=1}^{n}\frac{g^\varkappa}{f_\varkappa}(\bar{f}_\varkappa\bar{P}_{,\varkappa})_{,\varkappa} = g^i\bar{q}_i,$$

wo

$$q_\varkappa(x^\varkappa) \equiv \left[\frac{p''_\varkappa}{p_\varkappa} - 2\left(\frac{p'_\varkappa}{p_\varkappa}\right)^2 + \frac{p'_\varkappa f'_\varkappa}{p_\varkappa f_\varkappa}\right] + q_\varkappa.$$

Nach Wahl von $p_\varkappa = \sqrt{|f_\varkappa|}$ gilt mit $\bar{P} = P\prod_{\varkappa=1}^{n}\sqrt{|f_\varkappa|}$

$$\frac{1}{\bar{P}}\sum_{\varkappa=1}^{n} g^\varkappa\bar{P}_{,\varkappa\varkappa} = g^i\bar{q}_i. \tag{11}$$

Über die Funktionen $h_\varkappa(x^\varkappa)$ kann noch verfügt werden. Mit $h_\varkappa \equiv -(\log|f_\varkappa|)_{,\varkappa}$ folgt aus (10)

$$\bar{P}^2 = \sqrt{|g|}\exp\int^{(x)}\sum_{\varkappa=1}^{n}(\log|g^\varkappa|)_{,\varkappa}\,dx^\varkappa. \tag{12}$$

Durch Anwendung des Differentialoperators $L_{\varkappa\lambda}$ auf die Gleichung (11) folgt unter Beachtung des Lemmas 3, [2], (mit $Q := g^i q_i$) und der Relation (4)

$$L_{\eta\lambda}\left[\sum_{\nu=1}^{n} g^\nu\frac{\bar{P}_{,\nu\nu}}{\bar{P}}\right] = \sum_{\nu=1}^{\varkappa} g^\nu\frac{\bar{P}_{,\nu\nu}}{\bar{P}}f_{\varkappa\lambda}, \quad (\varkappa \neq \lambda). \tag{13}$$

Die Gleichungen (11) und (13) sind wegen Lemma 3, [2], äquivalent. Unter Berücksichtigung von (12) gilt

$$r_\varkappa := \frac{\overline{P}_{,\varkappa\varkappa}}{\overline{P}} \equiv \frac{1}{2}\left(\frac{1}{2}s_\varkappa^2 + s_{\varkappa,\varkappa}\right), \quad s_\varkappa \equiv [\log(\sqrt{|g|}\,|g^\varkappa|)]_{,\varkappa},$$

so dass (13) in der Form

$$\text{(C')} \qquad L_{\varkappa\lambda}[g^i r_i] = g^i r_i f_{\varkappa\lambda}, \quad (\varkappa \neq \lambda)$$

geschrieben werden kann. (C′) ist eine konkret nachprüfbare Bedingung.

Es gilt somit der

S a t z 2. *Die Laplace-Gleichung* (1) *ist in einem festen Koordinatensystem* $(x^1, \ldots, x^n)$ *des* V^n *genau dann P-separierbar, wenn die folgenden Bedingungen erfüllt sind:*

$$\text{(A)} \qquad g^{\varkappa\lambda} = g^\varkappa \delta^\varkappa_\lambda,$$

$$\text{(B'')} \qquad g_\mu L_{\varkappa\lambda}[g^\mu] = f_{\varkappa\lambda}, \quad (\varkappa \neq \lambda),$$

$$\text{(C')} \qquad L_{\varkappa\lambda}[g^i r_i] = g^i r_i f_{\varkappa\lambda}, \quad (\varkappa \neq \lambda),$$

mit $r_\varkappa(x) \equiv \frac{1}{2}\left[\frac{1}{2}s_\varkappa^2 + s_{\varkappa,\varkappa}\right]$; $s_\varkappa \equiv [\log(\sqrt{|g|}\,|g^\varkappa|)]_{,\varkappa}$.

Die Funktion $P(x)$ ist bis auf einen Produktfaktor gegeben durch

$$P^2 = \sqrt{|g|}\exp\left[\int^{(x)} \sum_{\varkappa=1}^{n} (\log|g^\varkappa|)_{,\varkappa}\, dx^\varkappa\right].$$

F o l g e r u n g 1. Aus (D′) folgt für $n = 2$ wegen $R_{\varkappa\lambda} = R_{\varkappa\lambda} = 0$, $(\varkappa \neq \lambda)$. (s. Gleichungen (5), (6)!) sofort $P = \prod_{\varkappa=1}^{n} p_\varkappa(x^\varkappa)$. Die Bedingungen (C) bzw. (C′) sind für alle derartigen Funktionen P erfüllt. D. h., die Laplace-Gleichung (1) ist im Falle $n = 2$ für alle konform-Stäckelschen Metriken vollständig separierbar.

F o l g e r u n g 2. Aus Satz 2 ergeben sich die Bedingungen für vollständige Separierbarkeit der Laplace-Gleichung (1) durch die Forderung $P = \prod_{\varkappa=1}^{n} p_\varkappa(x^\varkappa)$.

Die Bedingungen (A) und (B″) bleiben erhalten, (C′) ist erfüllt. Produktgestalt von P ist äquivalent mit $(\log P^2)_{,\varkappa}$ = Funktion von $x^\varkappa$ allein, also nach (12) mit

$$2\frac{P_{,\varkappa}}{P} \equiv [\log(\sqrt{|g|}\,|g^\varkappa|)]_{,\varkappa} \equiv s_\varkappa(x^\varkappa).$$

Diese Bedingung ist der von Steigenberger in [2] angegebenen Bedingung (C 6) äquivalent.

Der Satz 2 kann auf Separierbarkeitsprobleme folgendermassen angewendet werden: (A) und (B'') seien als notwendige Bedingungen für vollständige wie auch für P-Separierbarkeit erfüllt. Dann hat P entweder Produktgestalt, und die Gleichung (1) ist vollständig separierbar oder P hat nicht Produktgestalt, dann entscheidet Bedingung (C') über P-Separierbarkeit.

3. P-Separation der Schrödinger-Gleichung

Die bisherigen Überlegungen lassen sich übertragen auf die Untersuchung der P-Separierbarkeit der Schrödinger-Gleichung

$$\Delta U + K^2(E - V)U = 0; \tag{14}$$

K^2 ist eine feste Konstante, der Parameter $E =$ const. bezeichnet die Gesamtenergie und $V = V(x)$ das Potential.

Es gilt der

Satz 3. *Die Schrödinger-Gleichung* (14) *ist in einem festen Koordinatensystem* $(x^1, \ldots, x^n)$ *des* V^n *genau dann* P*-separierbar, wenn die folgenden Bedingungen erfüllt sind:*

$$g^{\varkappa\lambda} = g^{\varkappa}\delta^{\varkappa}_{\lambda}, \tag{$\bar{A}$}$$

$$L_{\varkappa\lambda}[g^{\mu}] = 0, \quad (\varkappa \neq \lambda), \tag{$\bar{B}$}$$

$$K^2 L_{\varkappa\lambda}[V] + L_{\varkappa\lambda}[g^i r_i] = 0, \quad (\varkappa \neq \lambda). \tag{$\bar{C}$}$$

Die Funktion P *wird nach* (12) *berechnet,* $r_{\varkappa}(x)$ *sind dieselben Funktionen wie in Satz 2.*

Diese Ergebnisse können dann auch auf beliebige homogene lineare partielle Differentialgleichungen zweiter Ordnung vom Typ

$$a^{ij}(x)U_{,ij} + b^i(x)U_{,i} + a(x)U = 0$$

angewendet werden. Die Matrix der a_{ij} sei regulär. Für die P-Separierbarkeit dieser Gleichung muss allerdings $b_i := a_{ij}b^j$ notwendig Gradient sein. (Das heisst z. B.: Die Schrödinger-Gleichung mit nichttrivialem Vektor-potential ist nicht P-separierbar).

4. Beispiele

4.1. Die Laplace-Gleichung

$$(pU_{,\lambda})_{,\lambda} + (pU_{,\alpha})_{,\alpha} + \frac{q}{p}U_{,\omega\omega} = 0$$

in einem euklidischen Raum E^3 mit orthogonalen rotationssymmetrischen Koordinaten λ, α, ω, $0 \leqslant \lambda \leqslant \pi$, $-\infty < \alpha < +\infty$, $0 \leqslant \omega \leqslant 2\pi$, für die der Zusammenhang mit den kartesischen Koordinaten durch

$$\pi x = \alpha - 1 + \exp(-\alpha) \cos \lambda,$$
$$\pi y = [\pi - \lambda + \exp(-\alpha) \sin \lambda] \sin \omega,$$
$$\pi z = [\pi - \lambda + \exp(-\alpha) \sin \lambda] \cos \omega$$

gegeben ist und wobei die Bezeichnungen

$$p(\lambda, \alpha) \equiv \pi - \lambda + \exp(-\alpha) \sin \lambda,$$
$$q(\lambda, \alpha) \equiv 1 - 2 \exp(-\alpha) \cos \lambda + \exp(-2\alpha)$$

verwendet wurden, ist nicht P-separierbar, da die Bedingung (B″) nicht erfüllt ist.

4.2. Die Laplace-Gleichung

$$\Delta U \equiv (e^{x^1} + e^{x^2})^{-6} \sum_{i=1}^{3} [(e^{x^1} + e^{x^2}) U_{,i}]_{,i} = 0$$

ist P-separierbar. Sie wurde von Ober-Kassebaum in der Arbeit [1] gemäss Satz 1 konstruiert. Die Metrik ist orthogonal. Die Bedingungen (B″) und (C′) sind erfüllt und es ist — bis auf einen Produktfaktor —

$$P = (e^{x^1} + e^{x^2}).$$

Diese Gleichung ist jedoch nicht vollständig separierbar. Der Separationsansatz

$$V(x) \equiv V_1(x^1) V_2(x^2) V_3(x^3) = (e^{x^1} + e^{x^2}) U$$

liefert die drei gewöhnlichen Differentialgleichungen

$$\begin{cases} V_1'' \qquad\quad - a_1 \; V_1 = 0, \\ V_2'' + (a_1 - a_2) \; V_2 = 0, \\ V_3'' + (a_2 - 1) \; V_3 = 0 \, ; \end{cases}$$

a_1 und a_2 sind Separationskonstanten.

4.3. Die Laplace-Gleichung sei in parabolischen Koordinaten $x^1 = u$, $x^2 = v$, $x^3 = \psi$, $0 \leqslant u < \infty$, $0 \leqslant v < \infty$, $0 \leqslant \psi < 2\pi$, die mit den kartesischen Koordinaten gemäss

$$x = uv \cos \psi, \quad y = uv \sin \psi, \quad z = \frac{1}{2} [u^2 - v^2]$$

zusammenhängen, auf Separierbarkeit zu untersuchen.

Die Metrik ist orthogonal und wegen $L_{\varkappa\lambda}[g^{\mu}] = 0$, $(\varkappa \neq \lambda)$, Stäckelsch. Die Bedingungen (A) und (B″) sind also erfüllt. Die Funktionen $s_{\varkappa}$ hängen jeweils nur von der gleichindizierten Variablen ab. Nach Folgerung 2 ist die Laplace-Gleichung in diesen Koordinaten vollständig separierbar.

Die gewöhnlichen Differentialgleichungen lauten

$$\begin{cases} U_1'' + \dfrac{1}{u} U_1' + \left(a_2 - \dfrac{1}{u^2} a_3\right) U_1 = 0, \\ U_2'' + \dfrac{1}{v} U_2' - \left(a_2 + \dfrac{1}{v^2} a_3\right) U_2 = 0, \\ U_3'' \qquad\qquad\qquad\qquad + a_3 U_3 = 0. \end{cases}$$

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UN CRITERIU PENTRU *P*-SEPARABILITATEA ECUAŢIEI LAPLACE ÎN SPAŢII RIEMANN

(Rezumat)

Prin analogie cu criteriul dat de Steigenberger [2] pentru separabilitatea generală a ecuaţiei lui Laplace în spaţii Riemann şi utilizînd unele rezultate ale lui Ober-Kassebaum [1], se stabileşte un criteriu care permite efectiv să se decidă dacă o ecuaţie Laplace dată este sau nu *P*-separabilă.

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SUR LES DISTRIBUTIONS NORMALES ET LEURS FONCTIONS CARACTÉRISTIQUES

PAR

VIOREL MURGESCU

1. Introduction

Dans [1] on a défini par une voie naturelle une suite de fonctions de répartition de la forme

$$\mathcal{F}_{(p)}(x)=\frac{p}{h_p}\Gamma\left(\frac{1}{2p}\right)^{-1}\int_{-\infty}^{x}\exp\left[-\left(\frac{t}{h_p}\right)^{2p}\right]\mathrm{d}t, \quad (p=1,2,\ldots), \tag{1.1}$$

qui proviennent d'une généralisation de la méthode des moindres carrés de Gauss (pour $p=1$, (1.1) est la fonction de répartition normale de Gauss-Laplace).

Les constantes h_p s'obtiennent d'habitude en fonction des données expérimentales, selon la loi

$$h_p \approx \mu_{(2p)}\sqrt[2p]{2p}, \quad (p=1,2,\ldots), \tag{1.2}$$

où $\mu_{(2p)}$ sont les valeurs empiriques des écarts moyens d'ordre $2p$ de la variable aléatoire $X(x)$.

Dans ce qui suit on se propose à présenter quelques propriétés des fonctions (1.1) et de leurs fonctions caractéristiques, de sorte que cette Note vient de compléter les résultats obtenues dans [1] et [2].

2. Observations sur les densités de probabilité

Les fonctions *densité de probabilité* qui correspondent aux lois normales ont l'expression

$$\rho_{(p)}^{(x)} = \frac{p}{h_p}\,\Gamma\left(\frac{1}{2p}\right)^{-1} \exp\left[-\left(\frac{x-\tilde{x}_{2p}}{h_p}\right)^{2p}\right], \quad (p = 1, 2, \ldots), \tag{2.1}$$

où $\tilde{x}_{2p}$ est la valeur moyenne d'ordre $2p$ d'une variable aléatoire $X(x)$, laquelle a été définie par la propriété de minimiser le moment centré d'ordre $2p$ correspondant et h_p est donné par (1.2).

On leur met en évidence les propriétés suivantes :

1) (2.1) sont des fonctions symétriques par rapport à $x = \tilde{x}_{2p}$, c'est-à-dire $\rho_{(p)}(\tilde{x}_{2p} - x) = \rho_{(p)}(x - \tilde{x}_{2p})$;

2) $\max\limits_{x \in R} \rho_{(p)}(x) = \rho_{(p)}(\tilde{x}_{2p}) = \left(\frac{p}{h_p}\right)\Gamma\left(\frac{1}{2p}\right)^{-1}, \quad (p = 1, 2, \ldots)$;

3) En supposant qu'il existe les limites de l'écart moyen d'ordre $2p$ et de la valeur moyenne du même ordre $\tilde{x}_{2p}$, pour $p \to \infty$, en notant alors

$$h_\infty = \lim_{p \to \infty} \mu_{(2p)} \sqrt[2p]{2p} = \lim_{p \to \infty} \mu_{(2p)},$$

$$\tilde{x}_\infty = \lim_{p \to \infty} \tilde{x}_{2p},$$

on peut définir la fonction-limite des densités de probabilité par la formule

$$\rho_{(\infty)}(x) \overset{\text{déf}}{=} \lim_{p \to \infty} \rho_{(p)}(x), \tag{2.2}$$

d'où il s'ensuit

$$\rho_\infty(x) = \begin{cases} \dfrac{1}{2h_\infty} & \text{pour} \quad |x - \tilde{x}_\infty| < h_\infty, \\[2ex] \dfrac{1}{2eh_\infty} & \text{pour} \quad |x - \tilde{x}_\infty| = h_\infty, \\[2ex] 0 & \text{pour} \quad |x - \tilde{x}_\infty| > h_\infty. \end{cases} \tag{2.2'}$$

Jusqu'à une transformation linéaire, cette fonction a un caractère asymptotique pour la classe des fonctions (2.1).

Une représentation graphique des cloches obtenues pour les premières valeurs de p pourrait suggérer la possibilité d'appliquer la loi normale généralisée aux problèmes particuliers de statistique. Pour réaliser cette représentation il suffit à considérer les fonctions réduites

$$\tilde{\rho}_{(p)}(x) = p\Gamma\left(\frac{1}{2p}\right)^{-1} \exp\left(-x^{2p}\right), \quad (p = 1, 2, \ldots),$$

et la fonction-limite

$$\widetilde{\rho}_\infty(x) = \begin{cases} \dfrac{1}{2} & \text{pour} \quad |x| < 1, \\[2ex] \dfrac{1}{2e} & \text{pour} \quad |x| = 1, \\[2ex] 0 & \text{pour} \quad |x| > 1. \end{cases}$$

3. Les fonctions de Gauss d'ordre p

Au but de calculer les probabilités de divers ordres d'une variable aléatoire (v. [1], p. 61), qui correspondent aux répartitions normales (1.1), il faut considérer les fonctions

$$\Theta_p(x) \overset{\text{déf}}{=} 2p\Gamma\left(\frac{1}{2p}\right)^{-1} \int_0^x \exp(-t^{2p})\,dt, \quad (p = 1, 2, \ldots), \tag{3.1}$$

qu'on nomme *fonctions de Gauss d'ordre p*

a) Si on développe la fonction $\exp(-t^{2p})$ en série potentielle (qui converge uniformément pour $|t| < +\infty$), on obtient *les expressions des fonctions $\Theta_p(x)$ en séries potentielles* uniformément convergentes pour $|x| < +\infty$:

$$\Theta_p(x) = 2p\Gamma\left(\frac{1}{2p}\right)^{-1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu x^{2p\nu+1}}{\nu!\,(2p\nu+1)}, \quad (p=1,2,..). \tag{3.2}$$

En particulier, pour $p = 1$, (3.2) devient

$$\Theta_1(x) = \frac{2}{\sqrt{\pi}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu x^{2\nu+1}}{\nu!\,(2\nu+1)} = \operatorname{erf} x.$$

b) De (3.1) il résulte immédiatement

$$\Theta_p(0) = 0\,; \quad \lim_{x\to+\infty} \Theta_p(x) = 1, \quad (p = 1, 2, \ldots), \tag{3.2'}$$

et par comparaison de quelques valeurs des fonctions $\Theta_p(x)$, pour $p = 1, 2, 3$, calculées avec les formules

$$\Theta_1(x) = \operatorname{erf} x, \quad \Theta_2(x) = 1{,}103\,265 \int_0^x \exp(-t^4)\,dt,$$

$$\Theta_3(x) = 1{,}077\,887 \int_0^x \exp(-t^6)\,dt,$$

à une erreur de 10^{-6}, on peut envisager une représentation approximative de ces fonctions donnée dans le tableau 1, et présentée dans la fig. 1.

Tableau 1

x	$\Theta_1(x) = \mathrm{erf}\,x$	$\Theta_2(x)$	$\Theta_3(x)$
0,0	0,00000	0,00000	0,00000
0,1	0,11246	0,11032	0,10778
0,2	0,22270	0,22058	0,21557
0,3	0,32863	0,33092	0,32333
0,4	0,42839	0,43904	0,43090
0,5	0,52050	0,54485	0,53774
0,6	0,60386	0,64139	0,64247
0,7	0,67780	0,73753	0,74224
0,8	0,74210	0,81775	0,85152
0,9	0,79691	0,88324	0,90568
1,00	0,84270	0,93207	0,95744

c) On peut donner encore *une représentation des fonctions de Gauss d'ordre p, $\Theta_p(x)$, par des fonctions hypergéométriques dégénérées.*

Pour ce but, on voit qu'en notant

$$[\lambda]_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)},$$

on peut vérifier immédiatement qu'en posant $\lambda = 1/(2p)$, on obtient

$$\frac{1}{2\nu + \dfrac{1}{p}} = p\,\frac{\left[\dfrac{1}{2p}\right]_\nu}{\left[\dfrac{1}{2p} + 1\right]_\nu}. \tag{3.3}$$

En substituant (3.3) dans (3.2), ces dernières deviennent

$$\Theta_p(x) = \frac{2x}{\Gamma\left(\frac{1}{2p}\right)} \sum_{\nu=0}^{\infty} \frac{(-x^{2p})^\nu}{\nu!\left(2\nu + \frac{1}{p}\right)} = \frac{2px}{\Gamma\left(\frac{1}{2p}\right)} \sum_{\nu=0}^{\infty} \frac{\left[\frac{1}{2p}\right]_\nu}{\nu!\left[\frac{1}{2p}+1\right]_\nu} (-x^{2p})^\nu,$$

ou

$$\Theta_p(x) = \frac{2px}{\Gamma\left(\frac{1}{2p}\right)} F\left(\frac{1}{2p}, \frac{1}{2p}+1, -x^{2p}\right), \quad (p = 1, 2, \ldots), \tag{3.4}$$

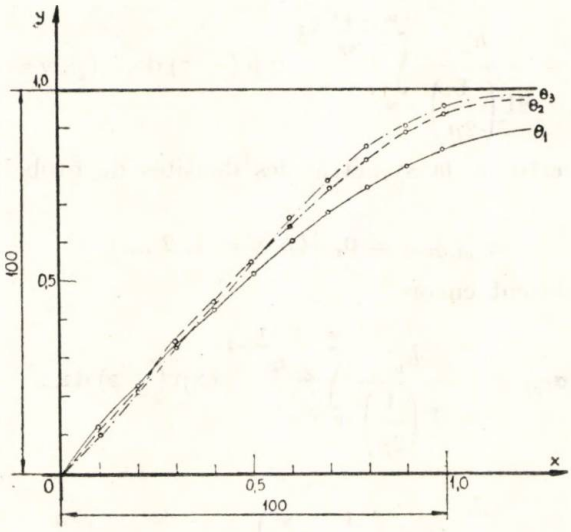

Fig. 1

ce qui représente les formules de liaison entre les fonctions $\Theta_p(x)$ et les fonctions hypergéométriques dégénérées [3], qui sont définies par les séries

$$F(\alpha, \gamma, z) \overset{\text{déf}}{=} \sum_{k=0}^{\infty} \frac{[\alpha]_k}{[\gamma]_k k!} z^k.$$

Pour $p = 1$ on obtient de (3.4) la formule déjà connue

$$\Theta_1(x) = \operatorname{erf} x = \frac{2x}{\sqrt{\pi}} F\left(\frac{1}{2}, \frac{3}{2}, -x^2\right).$$

On fait la remarque que les formules (3.1), (3.2) et (3.4) restent valables même dans le domaine complexe.

4. Le calcul des moments des distributions normales et leurs fonctions caractéristiques

Les moments $\alpha_{(p),\nu}$, $(p, \nu = 1, 2, \ldots)$, des distributions normales (1.1) ont l'expression

$$(4.1) \qquad \alpha_{(p),\nu} = \frac{p}{h_p}\Gamma\left(\frac{1}{2p}\right)^{-1}\int_{-\infty}^{+\infty} x^{\nu}\exp\left[-\left(\frac{x}{h_p}\right)^{2p}\right]dx, \quad (p, \nu = 1, 2, \ldots).$$

En faisant la substitution $x = h_p z^{\frac{1}{2p}}$, (4.1) prennent la forme

$$(4.2) \qquad \alpha_{(p),\nu} = \frac{h_p^{\nu}}{2\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{+\infty} z^{\frac{\nu+1}{2p}-1}\exp(-z)\,dz, \quad (p, \nu = 1, 2, \ldots).$$

Mais, en vertu de la symétrie des densités de probabilité, il résulte de (4.1)

$$(4.3) \qquad \alpha_{(p),2\nu-1} = 0, \quad (p, \nu = 1, 2, \ldots),$$

et de (4.2) on obtient encore

$$\alpha_{(p),2\nu} = \frac{h_p^{2\nu}}{\Gamma\left(\frac{1}{2p}\right)}\int_{0}^{\infty} z^{\frac{2\nu+1}{2p}-1}\exp(-z)\,dz,$$

ou

$$(4.4) \qquad \alpha_{(p),2\nu} = h_p^{2\nu}\frac{\Gamma\left(\frac{1}{2p}+\frac{\nu}{p}\right)}{\Gamma\left(\frac{1}{2p}\right)}, \quad (p, \nu = 1, 2, \ldots).$$

En utilisant la formule de multiplication [1]) pour la fonction $\Gamma(x)$, les moments (4.4) peuvent être écrits encore de la manière suivante :

$$(4.4') \qquad \alpha_{(p),2\nu} = \left(\frac{h_p^2}{2\pi}\right)^{\nu}(2\nu+1)^{\frac{2\nu-p+1}{2p}}\prod_{j=1}^{2\nu}\Gamma\left(\frac{1}{2p}+\frac{j}{2\nu+1}\right), \quad (p, \nu = 1, 2, \ldots).$$

---

1) $\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{j=0}^{n-1}\Gamma\left(x+\frac{j}{n}\right)$.

Dans le cas qu'on considère une variable réduite, les moments (4.2) deviennent

$$\alpha_{(p),\,2\nu} = \frac{\Gamma\left(\frac{1}{2p} + \frac{\nu}{p}\right)}{\Gamma\left(\frac{1}{2p}\right)}, \quad (p,\ \nu = 1,\ 2\ ,\ldots). \tag{4.5}$$

Maintenant, en substituant dans (1.1) $t = h_p u$, on peut attacher aux fonctions de répartition

$$\mathcal{F}_{(p)}(x) = p\,\Gamma\left(\frac{1}{2p}\right)^{-1} \int_{-\infty}^{x} \exp\left(-u^{2p}\right) \mathrm{d}u, \qquad (p = 1,\ 2\ ,\ldots),$$

les fonctions caractéristiques

$$f_{(p)}(t) \overset{\text{déf}}{=} p\,\Gamma\left(\frac{1}{2p}\right)^{-1} \int_{-\infty}^{+\infty} \exp\left(itx - x^{2p}\right) \mathrm{d}x, \quad (p = 1,\ 2\ ,\ldots), \tag{4.6}$$

lesquelles, en vertu de leur réalité, peuvent être mises sous la forme

$$f_{(p)}(t) = 2p\,\Gamma\left(\frac{1}{2p}\right)^{-1} \int_{0}^{\infty} \exp\left(-x^{2p}\right) \cos tx\, \mathrm{d}x, \quad (p = 1,\ 2\ ,\ldots). \tag{4.6'}$$

En développant la fonction $\cos tx$ en série potentielle et puis en l'intégrant terme à terme, les fonctions (4.6′) deviennent

$$f_{(p)}(t) = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} t^{2\nu}}{(2\nu)!} \left[\frac{2p}{\Gamma\left(\frac{1}{2p}\right)} \int_{0}^{\infty} x^{2\nu} \exp\left(-x^{2p}\right) \mathrm{d}x\right], \quad (p = 1,\ 2\ ,\ldots), \tag{4.6''}$$

où les parenthèses représentent justement les moments $\alpha_{(p),\,2\nu}$ donnés par (4.5). Par conséquent, les fonctions caractéristiques des distributions normales prennent la forme

$$f_{(p)}(t) = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\, \alpha_{(p),2\nu}}{\Gamma(2\nu + 1)}\, t^{2\nu}, \quad (p = 1,\ 2\ ,\ldots), \tag{4.7}$$

où $\alpha_{(p),\,2\nu}$ ont l'expression (4.5).

Si pour $p = 1$, de (4.7) on obtient $f_{(1)}(t) = \exp\left(-t^2/4\right)$, il est peu probable que pour $p = 2,\ 3\ ,\ldots$, les séries (4.7) pourraient être représentées par des fonctions connues. Cette impossibilité est justifiée par l'inexis-

tence d'une formule convenable de division pour la fonction $\Gamma(x)$, dont l'unique représentation peut être écrite dans la forme suivante :

$$\Gamma\left(\frac{2\nu+1}{2p}\right) = 2p\int_0^\infty t^{2\nu}\exp\left(-t^{2p}\right)\mathrm{d}t, \quad (\nu = 1, 2, \ldots).$$

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ASUPRA DISTRIBUŢIILOR NORMALE ŞI FUNCŢIILE LOR CARACTERISTICE

(Rezumat)

În această Notă se prezintă unele proprietăţi ale funcţilor de repartiţie normale (1.1), care au fost introduse de autor în [1] şi se studiază *funcţiile Gauss de ordinul p*, (3.1), notate $\Theta_{(p)}(x)$, precum şi funcţiile caracteristice corespunzătoare funcţiilor de repartiţie (1.1). În această ordine de idei, se dau reprezentările (3.2) şi (3.4) ale funcţiilor $\Theta_{(p)}(x)$, prima în serie de puteri, iar a doua cu ajutorul funcţiilor hipergeometrice degenerate.

Apoi sînt calculate momentele de diverse ordine ale unei variabile aleatoare continue în raport cu distribuţiile normale prezentate, care au expresia (4.3)–(4.4). În fine, se găsesc expresiile (4.7) în serii de puteri ale funcţiilor caracteristice corespunzătoare funcţiilor de repartiţie (1.1) scrise sub formă redusă.

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C.D. 518.12

UNE NOTE SUR L'ERREUR RELATIVE DE TRONCATURE D'ORDRE λ SUR DES OPÉRATIONS ÉLÉMENTAIRES

PAR

D. PREVOT

0. Introduction

En calcul matriciel ou automatique on ne peut avoir accés aux nombres exacts, hormis dans quelques cas particuliers, mais soit aux nombres tronqués, soit aux nombres arrondis. D'où si l'on fait abstraction des erreurs de méthode, il subsiste à la fin d'un processus de calcul numérique un reliquat plus ou moins important d'erreur.

Les méthodes habituelles d'analyse numérique se révèlent mal adaptées pour l'étude de ces erreurs, tandis que des méthodes probabilistes tout en étant plus souples, peuvent conduire à plus d'information, quant à la précision des résultats.

Dans cette étude nous avons précisé ce qui se passe au niveau des opérations élémentaires en particulier dans le cas de l'opération produit.

1. Généralités

1.1. Troncature d'ordre λ

Soit $x \in R^*$. En écriture flottante normalisée de base b, on peut écrire x sous la forme :

$$x = \eta\, 0,\ \xi_1 \xi_2 \ldots \xi_\lambda\, \xi_{\lambda+1} \ldots b^p$$

où

$$\eta = +1,\ b \text{ entier } \geqslant 2,\ p \in \mathbf{Z},\ \xi_1 \in \{1, \ldots, b-1\},\quad \xi_i \in \{0, 1, \ldots, b-1\}.$$

On appelle tronqué d'ordre λ de x la quantité

$$x^{*(\lambda)} = \eta\, 0,\ \xi_1\, \xi_2 \ldots \xi_\lambda\, b^p.$$

1.2. Erreur relative de troncature d'ordre λ

1.2.1. *Sur un nombre.* Par définition, on appelle erreur relative de troncature d'ordre λ sur le nombre x la quantité :

$$\varepsilon r(\lambda) = \frac{x - x^{*(\lambda)}}{x}. \tag{1}$$

1.2.2. *Sur une opération τ de degré p.* Soit τ un opérateur élémentaire : $./, +, -$; soient p opérandes non nuls : $x_1, x_2, \ldots, x_p$ et soient

$$y_1 = x_1 ; \; y_j = y_{j-1} \tau x_j, \quad (j = 2, 3, \ldots, p) ;$$

$$\widetilde{y}_1^{(\lambda)} = x_1^{*(\lambda)} = \widetilde{y}_j^{(\lambda)} (\widetilde{y}_{j-1}^{(\lambda)} + \tau x_j^{*(\lambda)})^{*(\lambda)}, \qquad (j = 2, 3, \ldots, p) ;$$

y_p est le résultat théorique, $\widetilde{y}_p^{(\lambda)}$ est le résultat obtenu. On appelle erreur relative de troncature d'ordre λ sur l'opération τ de degré p la quantité :

$$\varepsilon r(\lambda, p) = \frac{y_p - \widetilde{y}_p^{(\lambda)}}{y_p}. \tag{2}$$

Remarques. 1° $\varepsilon r(\lambda, 1) = \varepsilon r(\lambda)$;
2° pour l'étude de $\varepsilon r(\lambda)$ on peut supposer $x \in [b^{-1}, 1[$;
3° pour l'étude de $\varepsilon r(\lambda, p)$ on peut supposer $x_i \in [b^{-1}, 1[$ pour $\tau = .$ ou $\tau = /$

1.3. Hypothèses initiales

Tous les opérandes intervenant sont des réalisations de v. a. r. indépendantes, identiques de répartition uniforme sur $[b^{-1}, 1[$. Soit

$$X_i = \mathcal{U}_{([b^{-1}, 1)[} \; \forall_i ; \quad \text{les } X_i \text{ indépendantes.} \tag{3}$$

2. Étude de $\varepsilon R(\lambda)$

2.1. Étude par simulation

Pour $\lambda = 1, 2, \ldots, 7$ nous avons simulé des échantillons de 800 individus $\varepsilon r(\lambda)$ dans le cas où $b = 10$.

Il semble que l'on ait [1, p. 13, fig. D 20.1]

$$\overline{\varepsilon r}(\lambda) = C_1 \, 10^{-\lambda+1} ; \qquad s(\lambda) = C_2 \, 10^{-\lambda+1}. \tag{4}$$

2.2. Étude théorique approchée

2.2.1. *Hypothèses. Résultats*

$$\begin{cases} H_1 : X = \mathcal{U}_{([b^{-1}, 1[)} ; \\ H_2 : Y = X - X^{*(\lambda)} = \mathcal{U}_{([0, b^{-\lambda}[)} ; \\ H_3 : X, Y \text{ indépendantes.} \end{cases}$$

Remarques. 1° H_1 et H_2 sont équivalentes ;

2° H_3 est une hypothèse satisfaisante pour $b = 10$ dès que $\lambda \geqslant 3$.

On montre en réalité que X et Y sont asymptotiquement non corrélées. Sous ces hypothèses on obtient pour $U = Y/X$

$$(5)\qquad \begin{cases} u < 0 \Rightarrow f_U(\widehat{u}) = 0 \\ 0 \leqslant u < b^{-\lambda} \Rightarrow = \dfrac{b+1}{2} b^{\lambda-1} \\ b^{-\lambda} \leqslant u < b^{-\lambda+1} \Rightarrow = \dfrac{b^{-\lambda+1}}{2(b-1)} \dfrac{1}{u^2} - \dfrac{b^{\lambda-1}}{2(b-1)}, \\ b^{-\lambda} \leqslant u \Rightarrow = 0, \end{cases} \qquad \text{densité ;}$$

$$6)\qquad \widehat{E}(U) = \frac{\operatorname{Log} b}{2(b-1)} b^{-\lambda+1} = C_1 b^{-\lambda+1}, \qquad \text{moyenne ;}$$

$$(7)\qquad \widehat{\sigma}_U = \frac{1}{3b} - C_1^2 b^{-\lambda+1} = C_2 b^{-\lambda+1}, \qquad \text{écart-type.}$$

2.2.2. *Conclusion. Remarques.* Les résultats obtenus par l'étude théorique approchée sont excellents pour décrire la simulation.

On observe : 1° pour $\lambda = 1, 2$ [1, p. 20, fig. E 62.1 et 2] un décolage entre la courbe empirique et la courbe théorique ; mais H_3 est en défaut ici ;

2° pour $\lambda = 3, 4, 5$ [1, p. 20, fig. E 62. 3, 4 et 5] une bonne description de la courbe empirique ;

3° pour $\lambda = 6, 7$ [1, p. 20, fig. E 62. 6 et 7] un décalage qui met en cause la génération des nombres équirépartis.

2.3. Étude théorique stricte

Il s'est avéré nécessaire d'appuyer la formulation approchée sur des bases moins critiquables que l'hypothèse H_3.

La courbe d'erreur (1) [1, p. 23] est très complexe d'où l'expression très compliqué de la densité stricte. Dans [1, p. 52, fig. G 21.1] nous avons donnée la courbe f_U pour $b = 10$, $\lambda = 1$. Par exemple dans le cas où $b = 10$ pour

$\lambda = 1$ f_U est définie par 9 arcs de courbes sur [0, 1/2[

$\lambda = 2$ f_U est définie par 90 arcs de courbes sur [0, 1/101[

$\lambda = 3$ f_U est définie par 900 arcs de courbes sur [0, 1/1001[

. . . . . .

Pour la moyenne on trouve :

$$(8)\qquad E(U) = 1 - \left(\lambda + \frac{1}{b-1}\right) \log b + \frac{b^{-\lambda+1}}{b-1} \operatorname{Log} \frac{b^{\lambda!}}{b^{(\lambda-1)!}}.$$

2.4. Comparaision des 2 études

2.4.1. *Densités.* On définit une fonction minorante $m(u)$ de $f_U(u)$ et une fonction majorante $M(u)$ de $f_U(u)$.

On montre que

$$m \begin{cases} -\leqslant -\hat{f}_U - \leqslant - \\ -\leqslant -f_U - \leqslant - \end{cases} M$$

et que m, M, $\hat{f}_U$, f_U ont toutes les quatres la même limite quand $\lambda \to \infty$.

2.4.2. *Paramètres.* Si on effectue le développement en série entière de la moyenne et l'écart-type de la loi stricte, on trouve pour termes principaux la moyenne et l'écart-type de la loi approchée.

Alors dès que $\lambda \geqslant 3$, la loi approchée suffira pour décrire parfaitement la simulation (où $b = 10$).

2.5. Étude par simulation

On démontre que $\varepsilon R(\lambda) \xrightarrow{PS} \varepsilon R(\infty)$, $\varepsilon R(\lambda) \xrightarrow{P} \varepsilon R(\infty)$, $\varepsilon R(\lambda) \xrightarrow{L_2} \varepsilon R(\infty)$ avec $E(\varepsilon R(\infty)) = \mathrm{Var}\,(\varepsilon R(\infty)) = 0$. Mais on ne connaît pas le type de la loi limite.

3. Étude de $\varepsilon R(\lambda, p)$. Cas où $\tau =$.

3.1. Étude par simulation

Pour diverses valeurs de λ, pour $p \leqslant 50$ nous avons simulé des échantillons de 800 individus dans le cas ou $b = 10$.

On observe : 1° a λ fixé et p variable, une croissance quasi linéaire si $p \leqslant 50$ pour $\lambda \geqslant 3$, de la moyenne et de la variance statistiques [1, p. 74, fig 121.1 et 2] ;

2° à λ fixé et p fixé, une distribution quasi normale pour $p = 50$ et $\lambda = 4$, de l'échantillon [1, p. 98, fig. 533.1].

3.2. Étude théorique á λ fixé p variable

3.2.1. *Généralités.* Soit

$$z_p = \text{erreur totale sur le produit de degré } p = \varepsilon r(\lambda, p) = \frac{y_p - \tilde{y}_p}{y_p};$$

$$\varepsilon r_{(p)} = \text{erreur sur le } p^e \text{ facteur} = \frac{x_p - x_p^{*(\lambda)}}{x_p};$$

$$\varepsilon r_{(\widetilde{p-1}.p^\star)} = \text{erreur sur la } (p-1)^e \text{ opération produit} =$$

$$= \frac{\widetilde{y}^{(\lambda)}_{p-1}\, x^{\star(\lambda)}_p - (\widetilde{y}^{(\lambda)}_{p-1}\, x^{\star(\lambda)}_p)^{\star(\lambda)}}{\widetilde{y}^{(\lambda)}_{p-1}\, x^{\star(\lambda)}_p}.$$

On montre que

$$(9)\quad \begin{cases} (1-z_p) = (1-z_{p-1})(1-\varepsilon r_{(p)})(1-\varepsilon r_{(\widetilde{p-1}.p^\star)}), \\ (1-z_p) = \prod\limits_{i=1}^{p}(1-\varepsilon r_{(i)}) \prod\limits_{i=2}^{p}(1-\varepsilon r_{(\widetilde{i-1}.i^\star)}). \end{cases}$$

On montre que lorsque x_i parcourt $[b^{-1}, 1[$ i alors z_p parcourt une surface d'erreur très compliquée dans R^{p+1}. Dans [1, p. 120, fig. N 21.1 et 2] on a représenté de telles surfaces dans le cas $b = 2$, $p = 1$, $\lambda = 1$ et 2.

3.2.2. *Limite de Z_p quand $p \to \infty$.* Soit

$$\Omega = (R^\star)^\infty;$$

$$Z_p \text{ l'application } \begin{cases} \Omega \to R, \\ \omega \to z_p. \end{cases}$$

On montre que Z_p est une surjection de Ω sur $I_p = [0, t_p[\subset [0, 1[$ et de plus que $I_1 \subset I_2 \subset \ldots \subset I_p \subset \ldots \subset I_\infty = [0, 1[$; $\forall p \geqslant 1$, à Z_p on associe la tribu T_p sur Ω engendrée par la classe des images inverses des boréliens de I_p par la famille finie des applications $\{Z_q\}_{q \leqslant p}$. Et on définit de même, la tribu T_∞ sur Ω, engendrée par la classe des images inverses des boréliens de I_p par la famille des applications $\{Z_q\}_{q<\infty}$.

On démontre alors que $\{Z_p, T_p\}_{(p \geqslant 1)}$ est une sous-martingale sur l'espace de probabilité (Ω, T_∞, P), ce qui permet en appliquant des théorèmes sur les martingales de montrer que :

1° lim p. s. (Z_p) existe soit Z_∞ (c'est la distribution de Dirac en 1);

2° $Z_p \xrightarrow[L_2]{P} Z_\infty$;

3° $E(Z_\infty) = 1$, $\mathrm{Var}(Z_\infty) = 0$.

3.2.3. *Type de la loi limite de Z_p.* On démontre que $1_\Omega - Z_p$ est asymptotiquement logarithme normale de paramètres $\widetilde{m}_p$ et $\widetilde{\sigma}_p$ soit : $\mathcal{LN}(\widetilde{m}_p, \widetilde{\sigma}_p)$; $\widetilde{m}_p$ et $\widetilde{\sigma}_p$ sont 2 paramètres que nous n'avons pas encore déterminé théoriquement.

Lorsque p est assez grand pour que l'on soit presque à la limite alors Z_p est quasiment normale. Ce que l'on a observé par la simulation.

3.3. Étude théorique á p fixé λ variable

Par un procédé analogue au précédent on montre que $\varepsilon R(\lambda, p)$ a une limite quand $\lambda \to \infty$ et que : $\varepsilon R(\lambda, p) \xrightarrow{P.s., P, L_2} R(\infty, p)$. On montre enfin que $E(\varepsilon R(\infty, p)) = \mathrm{Var}\,(\varepsilon R(\infty, p)) = 0$.

Le type de la loi reste toutefois inconnu.

3.4. Synthèse des deux études précédentes

À chaque application $\varepsilon R(\lambda, p)$ pour λ entier $\geqslant 1$, p entier $\geqslant 1$, on associe la tribu $T_{\lambda, p}$ sur Ω, engendrée par la classe des images inverses des boréliens de I_p par la famille finie des applications $\{\varepsilon R(\mu, q)\}_{\mu \leqslant \lambda, q \leqslant p}$ et on définit la tribu dite de base : $T_{\omega, \infty}$ sur Ω, engendrée par la classe des images inverses de boréliens de I_p par la famille des applications

$$\varepsilon R(\mu, q)_{(\mu < \infty, q < \infty)}.$$

On peut alors montrer le

Théorème.

$$\{\varepsilon R(\lambda, p), T_{\lambda, p}\}_{(\lambda \geqslant 1, p \geqslant 1)} \tag{10}$$

a) *à λ fixé est une sous-martingale en p sur* $(\Omega, T_{\infty\infty}, P)$;

b) *à p fixé est une sur-martingale en λ sur* $(\Omega, T_{\infty\infty}, P)$.

On obtient les mêmes conclusions quant à l'existence des limites séparément en p et séparément en λ. Parler de $\varepsilon R(\infty, \infty)$ n'a pas de sens.

3.5. Applications du calcul de $n/$

Nous avons montré que les 20 décimales de 1 775/ données dans la table de Mrs Giannesini et Rouits ont un sens.

3.6. Applications au cas de la puissance

3.6.1. *Étude par simulation.* Cette étude est considérablement entravée par les possibilités relativement faibles de l'ordinateur.

3.6.2. *Étude théorique.* Soit $\Omega' = \{\omega \in \Omega\,;\, x_i = x_1, \forall i \geqslant 1\}$ = diagonale de Ω ; $Z'_p = Z_p \mid \Omega$.

Dans le cas présent les opérandes intervenant lors du calcul de $\widetilde{y}_p^{(\lambda)}$ sont la même réalisation d'une même v. a. r., si bien que dans l'étude directe de Z'_p interviennent des problèmes délicats de dépendance $\forall p \geqslant 1$, a Z'_p on associe la tribu T'_p sur Ω' définie comme la trace sur Ω' de la tribu T'_p. On définit de même une tribu de base T'_∞ comme la trace sur Ω' de T_∞. Sur l'espace mesurable (Ω', T'_∞) trace Ω' de l'espace mesurable (Ω, T_∞) il nous faut définir une probabilité P. On ne peut pas définir P' à partir

de la trace de P sur (Ω', T_∞) car on ignore si $P^\star(\Omega') \neq 0$. On est conduit à définir P' par $P' = P^\star T_1 / P^\star T_1(\Omega')$.

On montre que $P^\star T_1(\Omega') = 1$ d'où $P' = P^\star T_1$. On démontre à nouveau que $\{Z'_p, T'_p\}_{(p \geqslant 1)}$ est une sous-martingale $(\Omega', T'_{\infty, p'})$.

On montre que

$$(11) \qquad \begin{cases} Z'_p \xrightarrow{P.S.\ P'.\ L_1} Z'_\infty,\ Z'_\infty = p \cdot Z_\infty | \Omega' ; \\ E(Z'_\infty) = E(Z_\infty) = 1 ; \quad \mathrm{Var}\,(Z'_\infty) = \mathrm{Var}\,(Z_\infty) = 0. \end{cases}$$

Alors Z'_∞ a le même type de loi que Z_∞ et toute l'étude faite pour Z_p aura son application pour le cas d'une puissance.

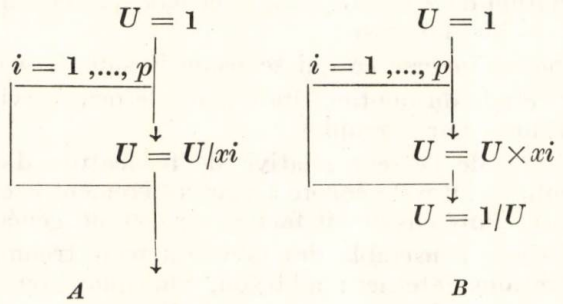
4. Étude de $\varepsilon R(\lambda, p)$. Cas où $\tau = /$

4.1. Étude par simulation

Lorsque $p \geqslant 2$, il arrive que l'on ait des $\varepsilon r(\lambda, p)_V < -1$ et il semble que la borne inférieure tende vers $-\infty$ quand $p \to +\infty$.

4.2. Étude théorique. Cas du quotient itéré

La conclusion que l'on peut tirer de cette étude, c'est qu'il faut remplacer chaque fois que cela est possible l'organigramme A par l'organigramme B



4.3. Étude théorique de l'erreur dans le cas de l'inverse

Soit

$$(12) \qquad U = \frac{1/x - (1/x^{\star(\lambda)})^{\star(\lambda)}}{1/x} ;$$

on montre que

$$(13)\quad f_U(u) = \frac{b^\lambda}{b-1} \sum_{k \in K(u)} 1/K \text{ avec } K(u), = \{k : u_1(k) \leqslant u < u_2(k)\} \cap \{b^{\lambda-1}, \ldots, b^\lambda\},$$

où

$$(14)\quad \begin{cases} u_1(k) = \left(\left\| \dfrac{b^{2\lambda-1}}{k+1} \right\| + 1 \right) b^{-2\lambda+1} k, \\ u_2(k) = \left(\left\| \dfrac{b^{2\lambda-1}}{k} \right\| + 1 \right) b^{-2\lambda+1} k; \end{cases}$$

$u_1(k)$ et $u_2(k)$ sont 2 fonctions très „capricieuses" [1, p. 154, fig. Q 32.1].

Un problème est ainsi posé, déterminer 2 fonctions $\hat{u}_1(k)$ et $\hat{u}_2(k)$ qui permettent de donner une estimation $\widehat{K}(u)$ de $K(u)$ donc de déterminer une loi approchée $\widehat{f_U}$ de f_U.

5. Conclusion

1. Cette étude soulève quelques problèmes non résolus :

a) détermination du type de loi limité en λ de $\varepsilon R(\lambda)$;

b) détermination d'un ordre optimal pour réaliser un produit de p nombres ;

c) détermination des paramètres théoriques m_p et σ_p ;

d) détermination du type de loi limité en λ de $\varepsilon R(\lambda, p)$ (produit) ;

e) détermination de 2 fonctions $\hat{u}_1(k)$ et $\hat{u}_2(k)$ permettant enfin d'achever l'étude du cas de l'inverse ;

f) étudier pour l'inverse ce qui se passe lorsque $\lambda \to \infty$;

g) achever l'étude du quotient itéré (car elle peut servir lors du calcul de fractions continues par exemple).

2. Pour l'étude de l'erreur relative de troncature d'ordre λ sur des opérations élémentaires, il reste encore à voir entièrement le cas d'une somme et d'une différence. Puis ensuite il faudra essayer de généraliser.

3. Si E_λ désigne l'ensemble des nombres réels tronqués, on peut le munir de 2 opérations internes : addition, multiplication. Toute la difficulté de l'étude provient du fait que ces opérations ne sont pas associatives.

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O NOTĂ ASUPRA ERORII DE TRUNCHIERE DE ORDIN λ ASUPRA OPERAŢIILOR ELEMENTARE

(Rezumat)

Datorită faptului că în calculul automat se operează de regulă cu numere rotunjite, în afara erorilor de metodă, la sfîrşitul unui proces de calcul numeric se înregistrează o anumită eroare, mai mult sau mai puţin importantă. Metodele uzuale ale analizei numerice s-au dovedit mai puţin eficiente pentru studiul acestor erori, în timp ce metodele probabiliste, mai suple, pot oferi mai multe informaţii.

Lucrarea analizează erorile ce se produc la nivelul operaţiilor elementare, în particular pentru cazul operaţiei produs.

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SUR LE PRINCIPE DU TRAVAIL VIRTUEL DANS LA DYNAMIQUE DES SYSTÈMES AYANT DE SOLIDES DE MASSES VARIABLES COMME ÉLÉMENTS CONSTITUTIFS

PAR

SPIRIDON DELEANU, N. IRIMICIUC et SILVIA NEAGU

0. Introduction

Tout en poursuivant la reconstruction de la mécanique analytique des corps de masses variables à l'aide d'un nouveau modèle mécanique qui intéresse quelques applications pratiques et qui a été présenté dans les travaux [1]...[3], — c'est-à-dire le solide dont la masse est distribuée en trois zones distinctes : une constante, la deuxième qui annexe et la troisième qui détache continuellement des particules matérielles — les auteurs présentent dans cette Note une nouvelle forme pour les équations d'équilibre dynamique de d'Alembert et pour la relation qui exprime le principe du travail virtuel.

I. Les équations d'équilibre dynamique de d'Alembert

On définit d'abord les composantes du torseur dans le pôle Q_ν (invariablement lié au solide de masse variable (S_ν) du système mécanique), des forces d'inertie de transport du solide (S_ν), les vecteurs déterminés par les relations :

$$
(\mathrm{I.1})\qquad \begin{cases} \overline{\mathcal{F}}^{i}_{\nu_t} = -\Big[\sum\limits_{(M_{c_\nu})} \mathrm{d}m\,\overline{a}_t + \sum\limits_{(M_{a_\nu})} \mathrm{d}m\,\overline{a}_t + \sum\limits_{(M_{d_\nu})} \mathrm{d}m\,\overline{a}_t\Big], \\ \overline{\mathcal{M}}^{i}_{Q_{\nu_t}} = -\Big[\sum\limits_{(M_{C_\nu})} \overline{r}' \times \mathrm{d}m\,\overline{a}_t + \sum\limits_{(M_{a_\nu})} \overline{r}' \times \mathrm{d}m\,\overline{a}_t + \sum\limits_{(M_{d_\nu})} \overline{r}' \times \mathrm{d}m\,\overline{a}_t\Big], \end{cases}
$$

les significations des symboles étant celles des travaux [1]...[3].

Tout en développant les termes on parvient aux expressions concrètes :

$$
\text{(I.2)}\quad \begin{cases}
\overline{\mathcal{F}}^{i}_{\nu_t} = -\left[\dfrac{d^{\star}}{dt}(\overline{\mathcal{H}}_{\nu_t}) - \left(\dfrac{dM_{\nu_1}}{dt}\overline{v}_{C_{a_{\nu_t}}} - \dfrac{dM_{\nu_2}}{dt}\overline{v}_{C_{d_{\nu_t}}}\right)\right], \\[2ex]
\overline{\mathcal{M}}^{i}_{Q_{\nu_t}} = -\left\{\dfrac{d^{\star}}{dt}(\overline{\mathcal{K}}_{Q_{\nu_t}}) - M_{\nu}(\overline{\omega}_{\nu} \times \overline{r}'_{C_{\nu}}) \times \overline{v}_{Q_{\nu}} - \right. \\[2ex]
\qquad \left. - \left[\dfrac{dM_{\nu_1}}{dt}\overline{r}'_{C_{a_{\nu}}} \times \overline{v}_{C_{a_{\nu_t}}} - \dfrac{dM_{\nu_2}}{dt}\overline{r}'_{C_{d_{\nu}}} \times \overline{v}_{C_{d_{\nu_t}}}\right]\right\},
\end{cases}
$$

qui permettent d'écrire les équations du mouvement du solide (S^{ν}) établies dans [3], sous la forme :

$$
\text{(I.3)}\quad \begin{cases}
\overline{\mathcal{F}}_{\nu} + \overline{\mathcal{F}}^{i}_{\nu_t} + \overline{\mathcal{F}}_{r_{\nu}} + \overline{\mathcal{R}}_{\nu} + \overline{\mathcal{R}}_{\text{int}_{\nu}} = 0, \\[2ex]
\overline{\mathcal{M}}_{Q_{\nu}} + \overline{\mathcal{M}}^{i}_{Q_{\nu_t}} + \overline{\mathcal{M}}_{Q_{\nu_r}} + \overline{\mathcal{M}}_{Q_{\nu_R}} + \overline{\mathcal{M}}_{Q_{\nu_{\text{int}}}} = 0.
\end{cases}
$$

Après le passage du pôle Q_{ν} au pôle fixe O, commun pour tous les solides du système mécanique, et après la sommation des équations des forces et des moments, on parvient aux équations :

$$
\text{(I.4)}\quad \begin{cases}
\sum\limits_{\nu}(\overline{\mathcal{F}}_{\nu} + \overline{\mathcal{F}}^{i}_{\nu_t} + \overline{\mathcal{F}}_{r_{\nu}}) + \sum\limits_{\nu}\overline{\mathcal{R}}_{\nu} = 0, \\[2ex]
\sum\limits_{\nu}(\overline{\mathcal{M}}_{\nu} + \overline{\mathcal{M}}^{i}_{\nu_t} + \overline{\mathcal{M}}_{\nu_r}) + \sum\limits_{\nu}\overline{\mathcal{M}}_{\nu_R} = 0,
\end{cases}
$$

qui représentent *les équations d'équilibre dynamique de d'Alembert pour un système mécanique* ayant des solides *de masses variables* comme éléments constitutifs.

II. Le principe du travail virtuel dans la dynamique des solides de masses variables

Après la multiplication scalaire des équations (I.3) par les déplacements virt els de transport $\delta_{\overline{r}_{Q_{\nu}}}$ et respectivement $\delta\theta_{\nu} = \delta\theta_{\nu_p}\overline{i}_3 + \delta\theta_{\nu_r}\overline{i}'_3 + \delta\theta_{\nu_n}\overline{n}^{(\nu)}$, après la sommation des relations ainsi multipliées et tout en considérant que les liaisons extérieures et intérieures sont idéales et appliquées seulement dans les zones de masses constantes, c'est-à-dire en écrivant

$$
\text{(II.1)}\quad \sum_{\nu}(\overline{\mathcal{R}}_{\nu}\,\delta\overline{r}_{Q_{\nu}} + \overline{\mathcal{M}}_{Q_{\nu_R}}\,\delta\overline{\theta}_{\nu}) = 0
$$

et

$$\sum_{\nu} \overline{\mathcal{R}}_{\mathrm{int}_\nu} \delta \overline{r}_{Q_\nu} + \overline{\mathcal{M}}_{Q_{\nu\,\mathrm{int}}} \delta \overline{\theta}_\nu) = 0, \tag{II.2}$$

on obtient la relation :

$$\sum_{\nu} [(\overline{\mathcal{F}}_\nu + \overline{\mathcal{F}}_{\nu_r} + \overline{\mathcal{F}}^i_\nu) \delta \overline{r}_{Q_\nu} + (\overline{\mathcal{M}}_{Q_\nu} + \overline{\mathcal{M}}_{Q_{\nu_r}} + \overline{\mathcal{M}}^i_{Q_{\nu_t}}) \delta \overline{\theta}_\nu] = 0, \tag{II.3}$$

qui représent *l'équation générale de la dynamique des systèmes mécaniques de masses variables.*

La dernière relation peut être écrite sous la forme :

$$\delta^\star \mathcal{L} + \delta^\star \mathcal{L}^i_t + \delta^\star \mathcal{L}_{r_t} = 0, \tag{II.4}$$

qui exprime *le principe du travail virtuel dans la dynamique des systèmes mécaniques de masses variables.* (L'indice (*) désigne qu'il faut considérer seulement les deplacements virtuels de transport et l'indice $(_t)$, indique que les forces respectives doivent être considérées seulement pour le mouvement de transport).

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ASUPRA PRINCIPIULUI DEPLASĂRILOR VIRTUALE ÎN DINAMICA SISTEMELOR AVÎND SOLIDE CU MASĂ VARIABILĂ CA ELEMENTE CONSTITUTIVE

(Rezumat)

Se stabilesc, în lucrare, noi forme pentru ecuaţiile de echilibru dinamic ale lui d'Alembert (I.1), (I.8), (I.4), precum şi pentru relaţia exprimînd principiul deplasărilor virtuale (II.3), (II.4), considerîndu-se sistemul mecanic constituit — nu din puncte cu masă variabilă, cum se procedează obişnuit —, ci din solide cu masă variabilă, concepute în sensul modelului prezentat în lucrările [1] ... [3], model care interesează unele aplicaţii tehnice.

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Secţia I
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

C.D. 531.012; 515.2

QUELQUES APPLICATIONS DE LA GÉOMÉTRIE REPRÉSENTATIVE PLANE-AXIELLE DANS LA CINÉMATIQUE GRAPHIQUE SPACIELLE

PAR

C. HUIU, ELISABETA RUSU et N. IRIMICIUC

0. Introduction

Les auteurs de cette Note, tout en fructifiant une idée plus ancienne concernant la représentation graphique des objets spaciels à l'aide de laquelle les uns d'entre eux, [2], [8], [10], ont élaboré la *géométrie représentative plane-axielle*, présentent, dans ce qui suit, une nouvelle méthode, tout simple et expéditive — évidemment, par rapport à d'autres méthodes —, pour la détermination graphique des configurations des points d'un solide rigide en mouvement spaciel.

1. Paramètres de position plan-axiels d'un solide rigide en mouvement spaciel

Le mouvement spaciel d'un solide rigide serra considéré déterminé par les équations finites habituelles :

$$(1) \qquad \begin{cases} x_{Q_\sigma} = x_{Q_\sigma}(t)\,; & (\sigma = 1, 2, 3)\,; \\ \theta_e = \theta_e(t)\,; & (e = p, r, n), \end{cases}$$

où x_{Q_σ} désignent les coordonnées par rapport à un repère cartésien triorthogonal fixe, $Ox_1x_2x_3$, du pôle Q invariablement lié au solide et θ_e désignent les angles d'Euler (fig. 1).

On introduit, dans le cadre de la méthode exposée, les *paramètres de position plan-axiels du solide*, définis par les relations :

$$
(2)\qquad \begin{cases}
r_Q = \sqrt{x_{Q_1}^2 + x_{Q_2}^2} = r_Q(t), \\
\varphi_Q = \operatorname{arc\,tg} \dfrac{x_{Q_2}}{x_{Q_1}} = \varphi_Q(t), \\
z_Q = x_{Q_3} = z_Q(t), \\
\psi = \theta_p = \psi(t), \\
\chi = \operatorname{arc\,tg}(\operatorname{tg}\theta_r \cos\theta_n) = \chi(t), \\
\lambda = \sin\theta_r \sin\theta_n = \lambda(t),
\end{cases}
$$

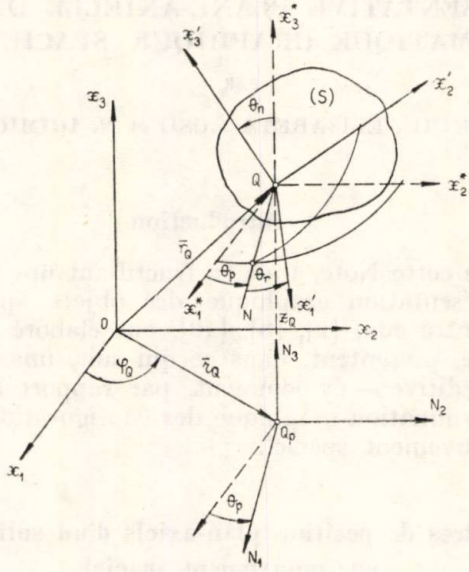

Fig. 1

r_Q, φ_Q, $z\varphi$ étant les coordonnées cylindriques du pôle Q, ψ — l'angle de précession, χ — la projection sur le plan $Ox_1^* x_2^*$, parallèle au plan $Qx_1 x_2$, de l'angle de rotation propre (fig. 2) et λ — le sinus de l'angle γ_1, formé par l'axe Qx_1' avec le plan $Qx_1^* x_2^*$ (fig. 3).

Les fig. 4...6 indiquent les constructions graphiques pour la détermination des paramètres χ et λ pour une valeur donnée de la variable t.

On introduit aussi, pour effectuer l'étude graphique du mouvement générale du solide, le *repère mobile plan-axiel* défini de la manière suivante :

1° l'origine, Q_p, du repère coïncide avec la projection sur le plan Ox_1x_2 du pôle Q invariablement lié au solide ;

2° l'axe Q_pN_1 coïncide avec la projection sur le plan Ox_1x_2 de la ligne des noeuds ;

3° l'axe Q_pN_3 a la même orientation comme l'axe Qx_3^* ;

4° l'axe Q_pN_2 complète le trièdre triorthogonal droit $Q_pN_1N_2N_3$.

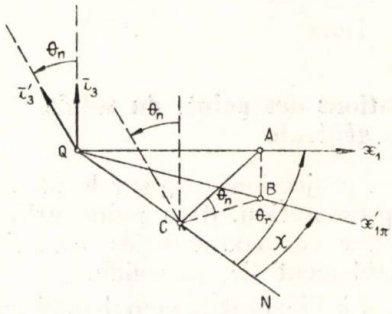

Fig. 2

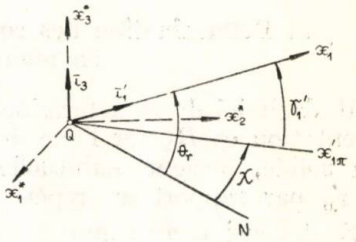

Fig. 3

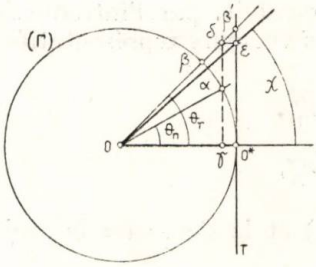

Fig. 4

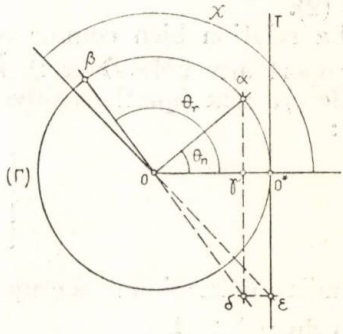

Fig. 5

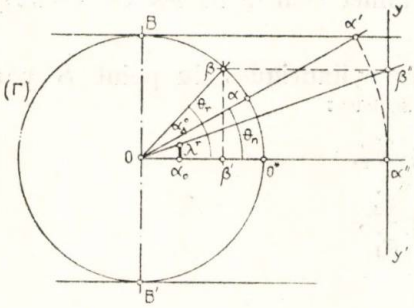

Fig. 6

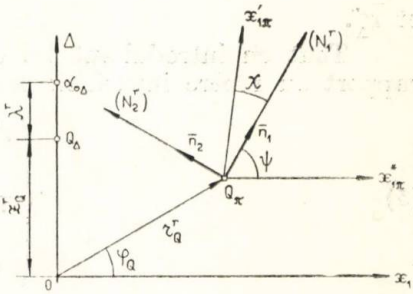

Fig. 7

Ce repère est figuré, à l'aide des conventions adoptés dans la géométrie représentative plane-axielle, dans la fig. 7 (le plan Ox_1x_2 coïncide avec le plan de la représentation, l'axe Ox_3, rabattue dans le plan Ox_1x_2,

coïncide avec l'axe de la représentation), toutes les distances étant représentées par des segments représentatifs notés avec l'indice supérieur (^{r}), à l'échelle :

$$\sigma_L = \frac{\nu_L}{q_L}, \quad \left[\frac{\mathrm{m}}{\mathrm{cm}}\right]. \tag{3}$$

2. Détermination des configurations des points du solide en mouvement générale

Il s'agit ici de la détermination des projections, B_Π sur le plan de la représentation et B_Δ sur l'axe de la représentation, d'un point arbitraire, B, du solide, désigné habituellement par les coordonnées cartesiennes x'_1, x'_2, x'_3, par rapport au repère invariablement lié au solide.

Sont considérées connues les valeurs à l'instant t, pour lequel on doit déterminer la position du point B, de tous les paramètres de position plan-axiels (2).

La relation bien connue entre les vecteurs de position du point B par rapport aux pôles O et Q, $\bar{r} = \bar{r}_Q + \bar{r}'$, conduit, par l'introduction de l'échelle σ_L, aux équations suivantes entre les vecteurs représentatifs plan-axiels :

$$\begin{cases} \bar{r}^r_\Pi = \bar{r}^r_{Q_\Pi} + \bar{r}'^r_\Pi, \\ \bar{r}^r_\Delta = \bar{r}^r_{Q_\Delta} + \bar{r}'^r_\Delta, \end{cases} \tag{4}$$

la première déterminant la projection $B_\Pi\,(\bar{r}^r_\Pi)$ et la deuxième la projection $B_\Delta(\bar{r}'^r_\Delta)$ du point B.

Dans les relations (4) on doit déterminer seulement les vecteurs $\bar{r}'^r_\Pi$ et $\bar{r}'^r_\Delta$.

Tout en introduisant les coordonnées cylindriques du point B par rapport au repère invariablement lié au solide :

$$\begin{cases} r' = \sqrt{x'^2_1 + x'^2_2}, \\ \varphi'_B = \operatorname{arc\,tg} \dfrac{x'_2}{x'_1}, \\ z' = x'_3, \end{cases} \tag{5}$$

on détermine les paramètres χ_B et λ_B spécifiques au point B (fig. 8), à l'aide des relations :

$$\chi_B = \operatorname{arc\,tg}\left[\operatorname{tg}(\theta_r + \varphi'_B)\cos\theta_n\right] \tag{6}$$

et

(7) $$\lambda_B = \sin \gamma'_B = \sin (\theta_r + \varphi'_B) \sin \theta_n$$

et par des constructions graphiques identiques à celles utilisées dans les fig. 4 et 6 pour la détermination des grandeurs χ et λ, en remplaçant seulement l'angle θ_r par l'angle $(\theta_r + \varphi'_B)$.

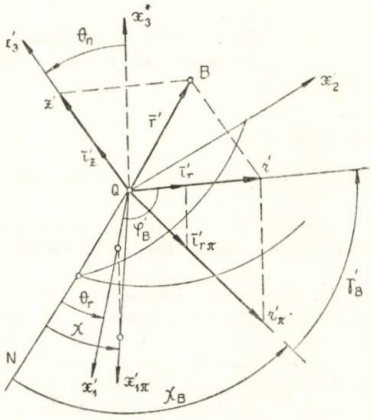

Fig. 8

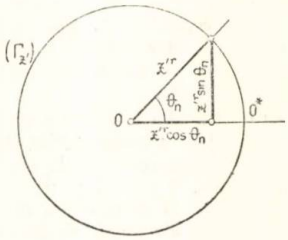

Fig. 9

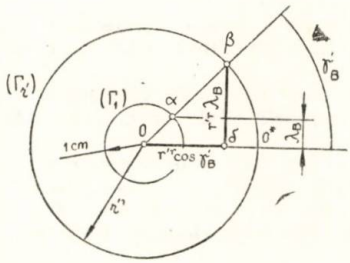

Fig. 10

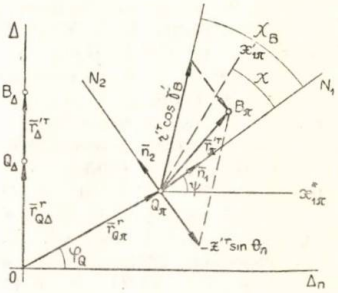

Fig. 11

On établit ainsi les relations :

(8) $$\begin{cases} \overline{r}'^{r}_{\Pi} = r'^{r} \cos \gamma'_B \, \overline{i}_{r\Pi} - z'^{r} \sin \theta_n \, \overline{n}_2 , \\ \overline{r}'^{r} = (r'^{r} \lambda_B + z'^{r} \cos \theta_n) \, \overline{i}_3 , \end{cases}$$

$\overline{i}_{r_\Pi}$ étant le vecteur unitaire de la composante orthogonale dans le plan $Qx_1^* x^*$ du vecteur :

(9) $$\overline{i}'_r = \cos \gamma'_B \, \overline{i}'_{r_\Pi} + \sin \gamma'_B \, \overline{i}_3 .$$

Les fig. 9 et 10 indiquent les constructions graphiques pour la détermination des différents termes qui entrent dans les relations (8).

Dans la fig. 11 sont indiquées, en fin, les constructions graphiques pour la détermination des projections B_Π et B_Δ du point B.

D'autres notes, qui seront publiées en futur, montreront les applications de la géométrie représentative plane-axielle pour la détermination graphique des distributions de vitesses et des distributions d'accélérations.

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UNELE APLICAŢII ALE GEOMETRIEI REPREZENTATIVE ÎN CINEMATICA GRAFICĂ SPAŢIALĂ

(Rezumat)

Introducîndu-se noţiunile de *parametri de poziţie plan-axiali* ai mişcării generale a solidului — definiţi prin relaţiile (2) — precum şi noţiunea de *reper mobil plan-axial*, se elaborează, pe baza ideilor de reprezentare grafică a obiectelor spaţiale expuse în *geometria reprezentativă plan-axială* creată de o parte dintre autori, o nouă metodă grafică de rezolvare a problemelor spaţiale de cinematică.

În nota de faţă sînt prezentate procedeele grafice pentru determinarea configuraţiilor punctelor unui solid în mişcare generală, determinare care are la bază relaţiile (4) şi (8).

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MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

C. Z. 512.2

CONTRIBUŢII LA TRASAREA LINIILOR DE INTERSECŢIE A CORPURILOR CARE AU BAZELE ÎN PLANE DIFERITE DE PROIECŢIE

DE

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1. Introducere

Intersecţiile de corpuri geometrice intervin în numeroase probleme aplicative. Grafica inginerească a elaborat procedee expeditive care să permită determinarea curbei de intersecţie dintre două suprafeţe date.

În lucrarea de faţă se prezintă un procedeu care permite determinarea porţiunilor văzută şi nevăzută ale liniei de intersecţie a două corpuri situate cu bazele în plane diferite de proiecţie, presupunînd soluţionată problema cînd bazele corpurilor sînt situate în acelaşi plan de proiecţie. În acest scop se utilizează, ca de obicei, plane secante pentru determinarea punctelor prin care se trasează curba sau poligonul de intersecţie.

2. Trasarea liniei de intersecţie

2.1. Intersecţia a două conuri circulare

Pentru a exemplifica procedeul care se propune, fie conurile S şi S_1 reprezentate în proiecţie ortogonală în fig. 1.

În primul rînd se cercetează genul intersecţiei, deci trebuie considerate planele limită. În cazul de faţă, pentru determinarea acestor plane, se duce dreapta (D) $[(d), (d'), (d'')]$ prin vîrfurile conurilor, găsindu-se urmele lor orizontală M şi laterală T. Apoi, prin urma laterală $T \equiv t''$ se construesc tangentele la cercul C (c, c', c''); interesează numai tangenta care trece prin $A \equiv a''$, care este urma laterală a planului determinat de dreapta (D) $[(d), (d'), (d'')]$ şi generatoarea SA. Se trasează în continuare urma orizontală L_{1h} a acestui plan, care în cazul studiat intersectează

cercul C_1 (c_1, c'_1, c''_1) în punctele *1* şi *2*. Este evident că planul joacă rol de plan limită.

Pentru obţinerea unui al doilea plan limită, se duc tangentele prin urma orizontală $M = m$ a dreptei (D) [(d), (d'), (d'')], la cercul C_1 (c_1, c'_1, c''_1). Este uşor de văzut că aici interesează planul L_2 a cărui urmă orizontală L_{2h} este tangentă la cercul C_1, urma sa laterală L_{2l} intersectînd cercul C (c, c', c'') în punctele *3* şi *4*.

După determinarea planelor L_1 şi L_2 se conchide că în cazul de faţă genul intersecţiei este o rupere.

Trebuie observat deci că în cazul corpurilor care au bazele în plane diferite de proiecţie, precizarea genului intersecţiei cere investigarea vizuală a întregului cîmp al desenului, trecînd de la un singur plan de proiecţie (cum se procedează la corpurile cu bazele în acelaşi plan de proiecţie), la două plane de proiecţie, în cazul considerat planele orizontal H şi lateral L.

Utilizînd planele limită şi urmărind epura se constată că sînt de acum obţinute puncte ale liniei de intersecţie, anume punctele α, α_1 — prin intermediul planului L_1 şi β, β_1 — prin intermediul planului L_2. Pentru a găsi alte puncte necesare trasării liniei de intersecţie, cuprinse între planele limită, se poate folosi orice plan secant. Aici s-au utilizat planele secante determinate de dreapta (D) [(d), (d'), (d'')] şi de generatoarele de contur aparent, anume planele P, P_1,...,P_4, prin intermediul cărora s au determinat o serie de puncte ale curbei de intersecţie.

Pentru a stabili ordinea de unire a punctelor, se recurge la metoda mobilului, considerat proiectat pe planele de proiecţie H şi L în care se găsesc bazele celor două conuri. Pentru a înlesni determinarea vizibilităţii punctelor şi stabilirea ordinii lor de unire, s-a întocmit tabelul 1, care ilustrează procedeul prezentat. Pe liniile 2 şi 3 ale tabelului sînt înscrise punctele în care se proiectează mobilul pe cercurile de bază atunci cînd el ajunge în planele secante indicate.

Se admite că în momentul iniţial mobilul se află în planul L_1, proiecţiile sale pe cele două baze fiind a'' pe planul lateral şi *1* pe planul orizontal. El se deplasează pînă ce proiecţiile sale ajung în f'' pe planul lateral de proiecţie şi în punctul *12* pe planul orizontal, ceea ce denotă că în acest moment mobilul se găseşte în planul P_2. În continuare mobilul trece prin d'' şi *5*, găsindu-se în respectivul moment în planul P.

Utilizarea tabelului 1 şi a celor două cercuri de bază permite precizarea rînd pe rînd a punctelor prin care trece mobilul. Pentru un control mai rapid al construcţiei s-a înscris pe linia 1 planul în care se găseşte mobilul corespunzător perechilor de proiecţii din liniile 2 şi 3 ale tabelului.

Spre deosebire de cazul cînd corpurile au bazele în acelaşi plan de proiecţie, aici sînt înregistrate pentru conul S — proiecţia laterală a mobilului (linia 2), iar pentru conul S_1 — proiecţia orizontală a mobilului (linia 3).

Linia a 4-a a tabelului indică ordinea de unire a punctelor, permiţînd urmărirea pe acelaşi plan de proiecţie a generatoarelor de intersecţie

Tabelul 1

Plane ajutătoare	L_1	P_2	P	P_4	P_3	P_1	L_2	P_1	P_3	P_4	P	P_2	L_1	P_2	P	P_4	P_3	P_1	L_2	P_1	P_3	P_4	P	P_2	L_1
Conul S	a''	$''$	d''	$19''$	$15''$	$''$	$4''$	$9''$	$15''$	$19''$	d''	f''	a''	$11''$	$7''$	$18''$	$16''$	$10''$	$3''$	$10''$	$16''$	$18''$	$7''$	$11''$	a''
Conul S_1	1	12	5	i	14	8	b	e	g	17	6	13	2	13	6	17	g	e	b	8	14	i	5	12	1
Ordinea de unire a punctelor	α	ε	γ	η	ζ	δ	β_1	δ_3	ζ_3	η_3	γ_3	ε_3	α_1	ε_2	γ_2	η_2	ζ_2	δ_2	β	δ_1	ζ_1	η_1	γ_1	ε_1	α
Vizibilitatea punctelor în proiecţie orizontală	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	v	v	v	v	v	v	v	v	n
Vizibilitatea punctelor în proiecţie verticală	n	n	n	v	v	v	v	v	v	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n
Vizibilitatea punctelor în proiecţie laterală	v	v	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	v	v	v	v	v	v	v	v

Tabelul 2

Plane ajutătoare	L_1	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	L_2	P_8	P_7	P_6	P_5	P_4	P_3	P_2	P_1	L_1	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	L_2	P_8	P_7	P_6	P_5	P_4	P_3	P_2	P_1	L_1
Conul C_1S_1	1	5	9	13	17	21	25	29	33	B	34	30	26	22	18	14	10	6	2	6	10	14	18	22	26	30	34	B	33	29	25	21	17	13	9	5	1
Conul CS	A	7	11	15	19	23	27	31	35	3	35	31	27	23	19	15	11	7	A	8	12	16	20	24	28	32	36	4	36	32	28	24	20	16	12	8	A
Ordinea de unire a punctelor	α_1	δ_1	γ_1	μ_1	ρ_1	λ_1	η_1	ε_1	ζ_1	β_1	ζ_2	ε_2	η_2	λ_2	ρ_2	μ_2	γ_2	δ_2	α_2	δ_3	γ_3	μ_3	ρ_3	λ_3	η_3	ε_3	ζ_3	β_2	ζ_4	ε_4	η_4	λ_4	ρ_4	μ_4	γ_4	δ_4	α_1
Vizibilitatea	n	n	n	n	n	n	n	v	v	v	v	v	n	n	n	n	n	n	n	n	n	n	n	n	v	v	v	v	v	v	v	v	v	v	v	n	n

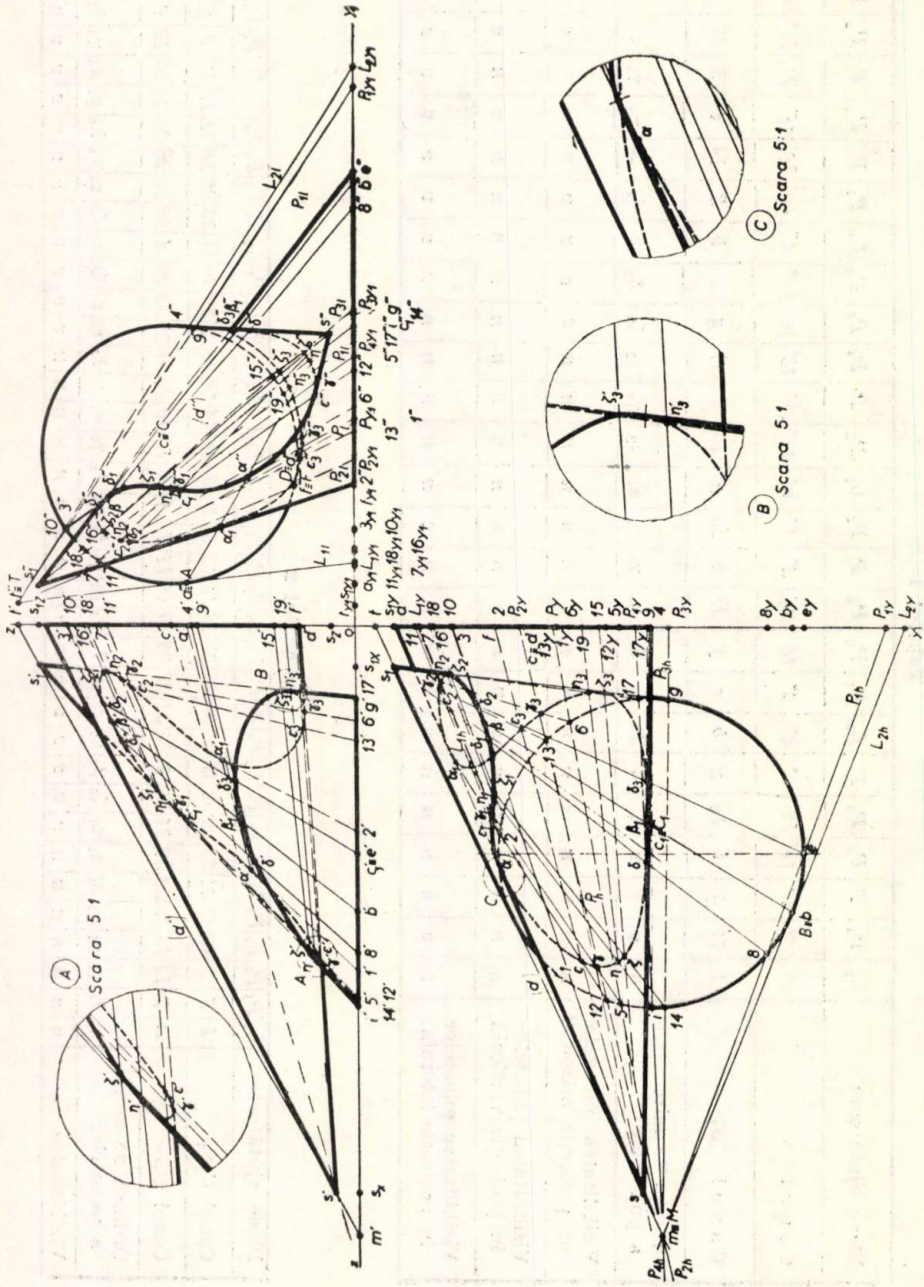

Fig. 1

sau de tangenţă a conurilor cu planele secante. De exemplu, în planul L_1 este notată proiecţia laterală a'' şi proiecţia orizontală *1*. Urmărind proiecţia laterală $s''a''$ a generatoarei de tangenţă a planului L_1 şi proiecţia laterală $s_1''1''$ a generatoarei după care planul L_1 intersectează conul S_1, se deduce că intersecţia lor este punctul α'' (proiecţia laterală a mobilului); urmărind aceleaşi generatoare în proiecţie verticală, la intersecţia lor se obţine α', proiecţia verticală a mobilului în momentul cînd se găseşte în planul L_1; în mod similar, proiecţiile orizontale *sa* şi $s_1 1$ dau la intersecţia lor proiecţia orizontală α a mobilului.

În legătură cu completarea liniei 4 a tabelului trebuie observat că aici poate fi trecută oricare din proiecţiile mobilului, deoarece ordinea de unire a punctelor este aceeaşi în fiecare proiecţie. S-a convenit să se noteze cu α proiecţia orizontală, trecută în linia a 4-a, coloana lui L_1, a'' şi *1*. Raţionînd în mod analog atunci cînd mobilul ajunge în planul P_2, coloana lui P_2, f'' şi *12*, se obţine punctul ε ş. a. m. d.

Vizibilitatea a fost înregistrată în tabelul 1, pentru fiecare plan de proiecţie în parte, notînd prescurtat vizibilitatea punctelor văzute prin *v*, şi a celor nevăzute prin *n*.

Vizibilitatea curbei de intersecţie se stabileşte ţinînd seama de faptul că *dacă două puncte succesive sînt văzute, linia care le uneşte este văzută, iar dacă unul sau amîndouă punctele sînt nevăzute, linia care le uneşte este nevăzută* şi urmează de figurat cu o trăsătură întreruptă.

În liniile 5, 6 şi 7 ale tabelului 1 sînt înscrise respectiv vizibilităţile pentru proiecţiile orizontală, verticală şi laterală, care în general nu sînt identice, deoarece puncte care se văd, de exemplu în proiecţie laterală, pot să nu fie văzute în celelalte proiecţii şi invers.

După cum s-a mai precizat, proiecţiile orizontale ale generatoarelor *SA* şi $S_1 1$ se intersectează în α. Cum în această proiecţie generatoarea *SA* nu se vede, înseamnă că α este nevăzut; în consecinţă s-a înscris simbolul *n* pe coloana a'', *1*, α, linia a 5-a (v. detaliul *C*, fig. 1). În continuare punctele sînt nevăzute pînă la η_2 şi văzute pînă la ε_1.

În mod similar se determină vizibilitatea celorlalte puncte din proiecţia orizontală, precum şi a celor din proiecţiile verticală şi laterală.

Pentru a studia problema vizibilităţii în cazul reprezentării axonometrice, unde intervin alte generatoare de contur aparent decît în proiecţiile pe două plane sau trei plane, s-au reprezentat conurile *S* şi S_1 în proiecţie axonometrică izometrică mărită $K=m=n$ (fig. 2). Vizibilitatea a fost stabilită cu ajutorul procedeului prezentat, rezultatele fiind consemnate în tabelul 2.

2.2. Intersecţia a două poliedre

Fie prisma cu baza *ABCDEF* în planul orizontal de proiecţie avînd direcţiile muchiilor laterale AA_1 şi prisma cu baza *GHIJ* în planul lateral de proiecţie şi direcţia muchiilor laterale GG_1, reprezentate în proiecţie axonometrică ortogonală dimetrică mărită $K = 2m = n$ (fig. 3).

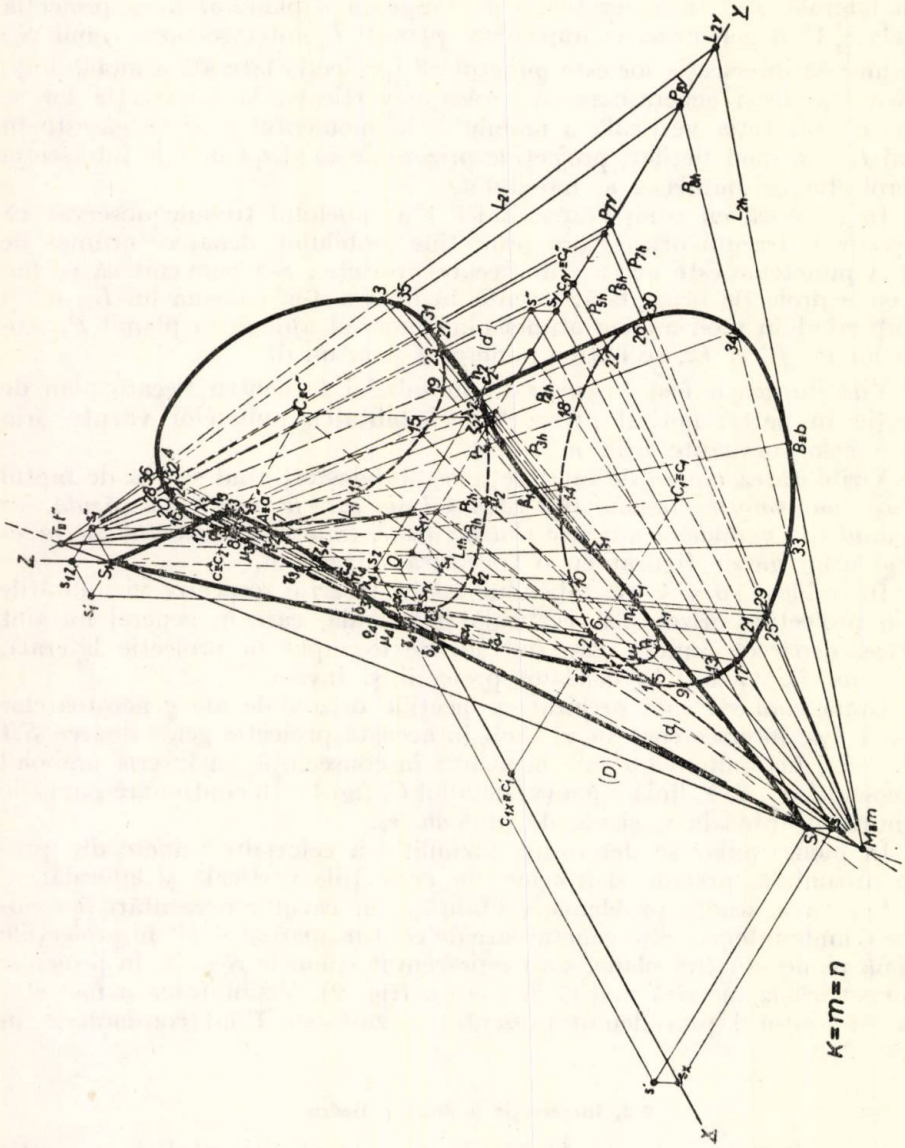

Fig. 2

După cum se ştie, poligonul de intersecţie se determină ducînd plane paralele cu muchiile ambelor prisme. Utilizînd metoda mobilului, se determină vîrfurile poligonului de intersecţie, care sînt consemnate în tabelul 3, ceea ce permite şi determinarea vizibilităţii punctelor. De exemplu, punctul γ_1 este determinat de muchia AA_1, care este văzută, şi de dreapta care trece prin punctul *5*, paralelă cu GG_1, şi care este nevăzută ; deci punctul γ_1 este nevăzut şi de aceea s-a înscris în ultima linie a tabelului 3 simbolul *n*. În mod similar s-a găsit că punctele β_1, ε_1 şi δ_2 sînt nevăzute.

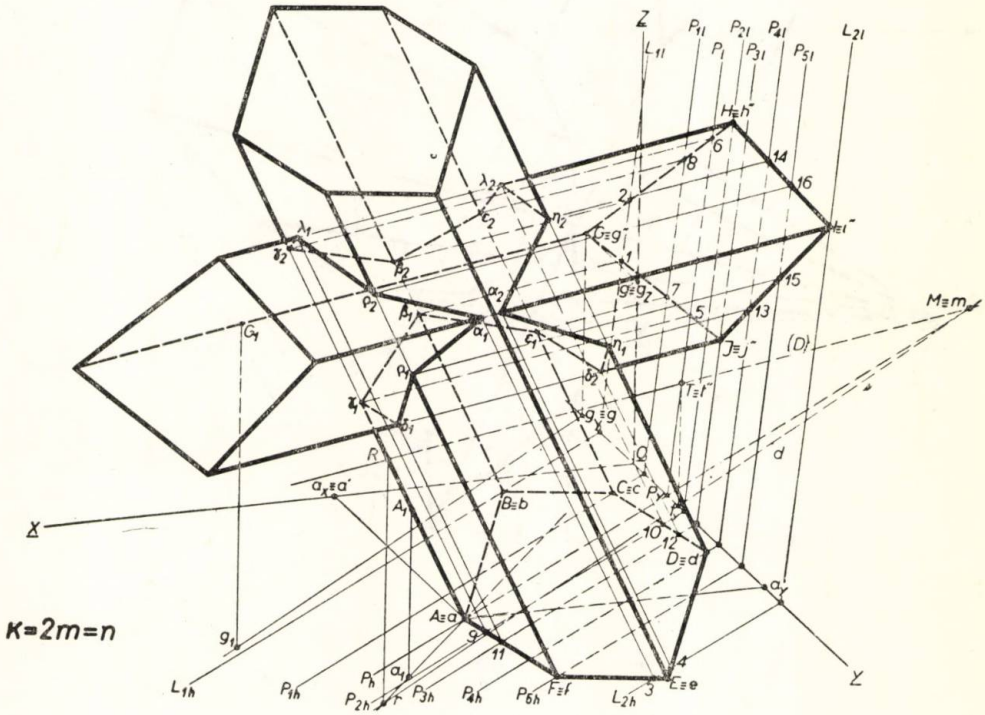

Fig. 3

Punctul η_1 este determinat de muchia DD_1 care este văzută şi de dreapta care trece prin punctul *13* paralelă cu GG_1 şi care este de asemenea văzută ; deci punctul η_1 este văzut. În continuare, punctele α_2, η_2 sînt văzute, λ_2, ε_2, β_2, γ_2 sînt nevăzute iar punctele λ_1, ρ_2, α_1, ρ_1, δ_1 sînt văzute. Ordinea de unire a vîrfurilor poligonului de intersecţie rezultă de asemenea din tabelul 3.

Observaţie. Regula enunţată anterior care preciza că linia care uneşte două puncte succesive văzute este văzută, prezintă în cazul poliedrelor următoarea excepţie : *dacă în şirul de puncte văzute care reprezintă vîrfu-*

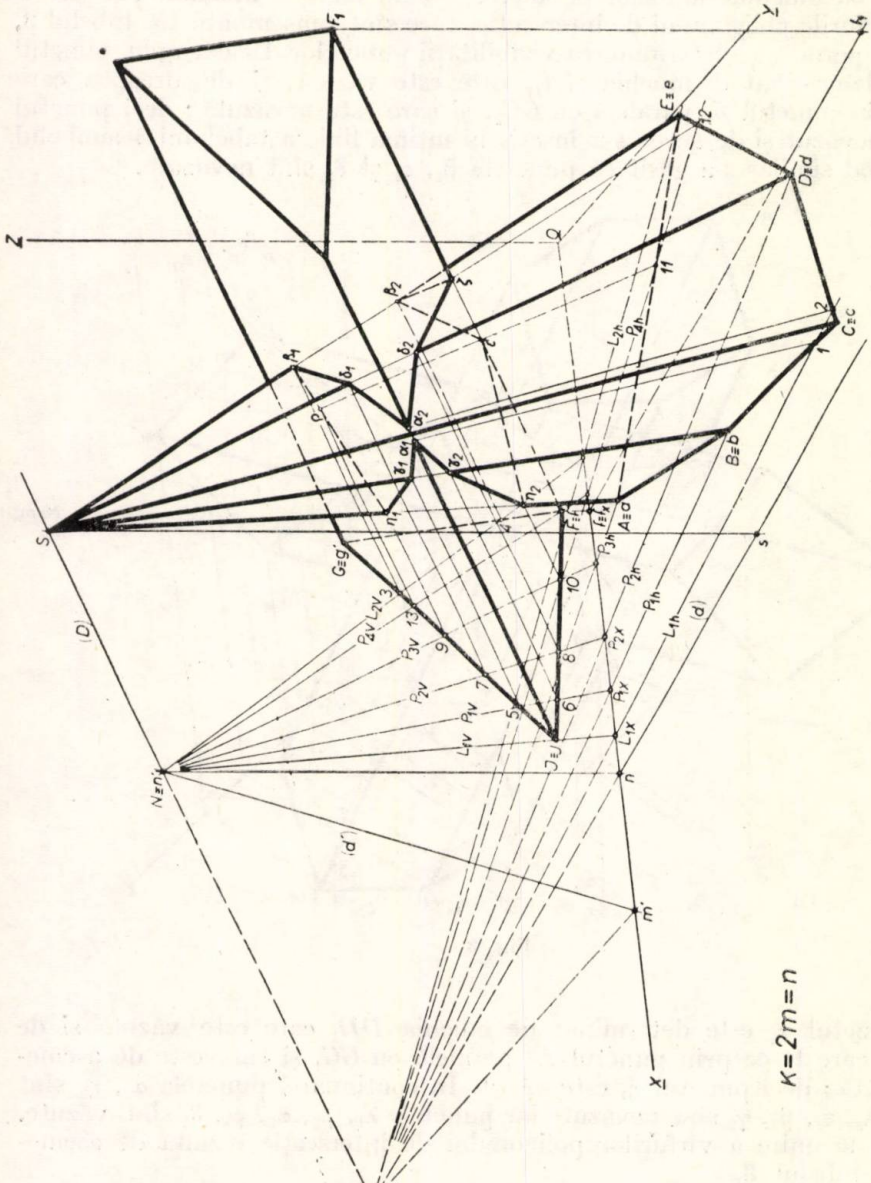

Fig. 4

rile poligonului de intersecţie se găsesc două puncte succesive care aparţin conturului aparent al aceluiaşi poliedru, latura determinată de cele două puncte este nevăzută.

Pentru a ilustra observaţia de mai sus, fie intersecţia dintre piramida $SABCDE$ cu baza în planul orizontal de proiecţie şi prisma $FGJJ_1F_1G_1$ cu baza în planul vertical de proiecţie (fig. 4). Pentru determinarea poligonului de intersecţie s-au utilizat plane duse prin dreapta (D) $[(d), (d')]$, ce trece prin vîrful S al piramidei şi este paralelă cu muchiile laterale ale prismei. Cele două vîrfuri ale poligonului de intersecţie η_1 şi β_1 sînt văzute; în conformitate cu tabelul 4 ar urma ca linia ce le uneşte să fie tot văzută, ceea ce nu corespunde realităţii (fig. 4). În baza observaţiei, vîrfurile η_1 şi β_1 fiind puncte văzute ce aparţin conturului aparent al piramidei, linia ce le uneşte este nevăzută, fapt ce corespunde evident realităţii.

Tabelul 3

Plane ajutătoare	P	L_1	P_1	P_2	P_4	L_2	P_4	P_2	P_1	L_1	P	P_2	P_5	L_2	P_5	P_3	P
Prisma cu baza $ABCDEF$	A	B	C	12	D	4	D	10	C	B	A	9	F	3	F	11	A
Prisma cu baza $GHIJ$	5	1	7	J	13	I	14	H	8	2	6	H	16	I	15	J	5
Ordinea de unire a punctelor	γ_1	β_1	ε_1	δ_2	η_1	α_2	η_2	λ_2	ε_2	β_2	γ_2	λ_1	ρ_2	α_1	ρ_1	δ_1	γ_1
Vizibilitatea	n	n	n	n	v	v	v	n	n	n	n	v	v	v	v	v	n

Tabelul 4

Plane ajutătoare	L_1	P_1	P_3	L_2	P_2	L_1	P_2	P_4	L_2	P_4	P_3	P_1	L_1
Piramida $SABCDE$	1	B	A	E	D	2	D	12	E	11	A	B	1
Prisma $FGJJ_1F_1G_1$	J	5	9	3	7	J	8	F	4	F	10	6	J
Ordinea de unire a punctelor	α_1	γ_1	η_1	β_1	δ_1	α_2	δ_2	ζ	β_2	ε	η_2	γ_2	α_1
Vizibilitatea	v	v	v	v	v	v	v	v	n	n	v	v	v

Pentru verificare grafică s-a folosit planul P_4 cu ajutorul căruia s-au determinat şi punctele suplimentare ε şi ζ. Acest plan determină pe latura $\eta_1\beta_1$ a poligonului punctul ρ, care este nevăzut, deci şi latura respectivă este nevăzută.

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CONTRIBUTIONS À LA DÉTERMINATION DE LA LIGNE D'INTERSECTION DES CORPS AYANT LES BASES DANS DIFFÉRENTS PLANS DE PROJECTION

(Résumé)

On présente une méthode pratique qui permet à déterminer la ligne d'intersection des corps ayant les bases dans des différents plans de projection. La méthode permet de résoudre d'une manière assez simple le problème de la visibilité.

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UMBRA CORPURILOR PRIN CORESPONDENŢĂ DE OMOLOGIE AFINĂ [1])

DE

I. RUSU

1. Introducere

Pentru a construi umbra corpurilor prin corespondenţă de omologie afină, se consideră că sursa de lumină este situată la infinit şi deci razele de lumină necesare formează cilindrul de lumină proiectant. Direcţia razei de lumină $R(r, R_0)$ este o dreaptă de poziţie generală.

Construcţia umbrei purtate şi proprii a unui corp geometric poate fi realizată considerîndu-se corespondenţa de omologie afină ce există între feţele corpului şi umbrele lor pe planele unui diedru de referinţă de unghi oarecare.

2. Umbra unei prisme

Fie prisma $ABCKMNPR$ pătrată dreaptă cu baza $ABCK$ situată în planul orizontal H al reprezentării (fig. 1). Se reprezintă raza de lumină $R(r, R_0)$, ce trece prin vîrful $M(m\,;\,5)$ al poliedrului dat. Pe planul H, baza $ABCK$ este şi propria sa umbră adică: $A_hB_hC_hK_h \equiv abck$. Umbra aruncată pe acelaşi plan H este limitată de proiecţiile razelor de lumină corespunzătoare vîrfurilor de pe muchiile laterale MA (ma) şi PC (pc). Din intersecţia acestora cu axa Ox rezultă punctele L şi L_1, ce determină linia de frîngere a umbrei.

Se trece apoi la construcţia umbrelor bazei superioare $MNPR$ pe planul H cît şi pe planul P. În acest scop trebuie intersectată raza de lu-

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[1]) Prezentată la sesiunea ştiinţifică a Institutului politehnic din Iaşi, din 2–4 iulie 1970.

mină $R(r, R_0)$, de exemplu aceea care trece prin vîrful $M(m\,;\,5)$, cu planele H şi P. Se consideră planul vertical dus prin raza R. Se rabate acest plan pe planul reprezentării împreună cu elementele conţinute. Se observă că R_0 (raza rabătută) intersectează planul H în m_0 deoarece $m_0 = R_0 \cap r$.

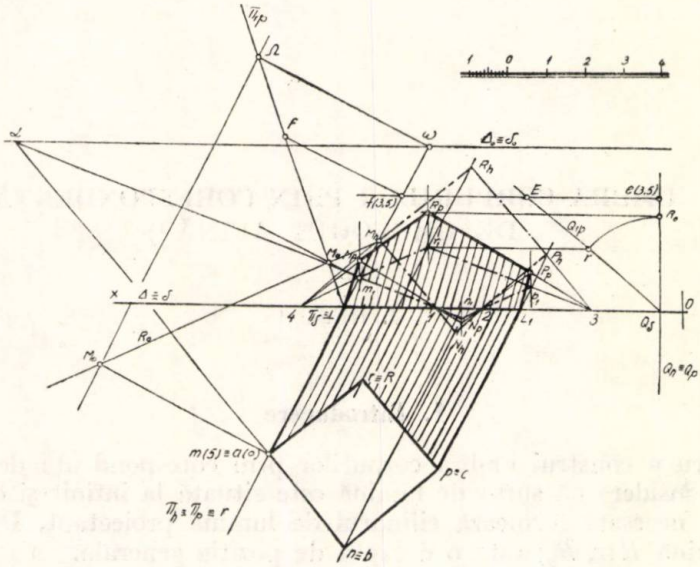

Fig. 1

Pătratul $MNPK$ din planul de nivel de cotă 5 are ca umbră pe planul H pătratul $m_0n_0p_0k_0$, congruent cu el. De aceea din proiecţia (umbra) m_0 se duc paralele la laturile concurente în $M(m)$, obţinîndu-se la intersecţia cu proiecţiile corespunzătoare ale razelor de lumină umbrele n_0 şi k_0 din care apoi se duc paralele la muchiile concurente în vîrful $P(p)$. Pătratul $m_0n_0p_0k_0$ este umbra bazei superioare a prismei pe planul reprezentării H.

Deci, umbra pe planul H a prismei date este $A_hB_hC_hp_0k_0m_0$ din care $LA_hB_hC_hL_1$ este reală, iar $Lm_0k_0p_0L_1$ este virtuală.

Umbra aceleiaşi baze $MNPK$ pe planul P se găseşte considerînd fie corespondenţa de afinitate dintre aceasta şi umbra ei, fie corespondenţa tot de afinitate între umbra pe planul H, care este construită, şi umbra pe planul P ce trebuie găsită.

În poziţia rabătută raza de lumină intersectează urma $\Pi_{1p} = \Pi \cap P$ în punctul M_1, care este umbra vîrfului M pe planul P. Se proiectează ortogonal acest punct M_1 pe planul H, adică: $(M_1\, m_1) \perp r \equiv \Pi_h$. Se observă că laturile pătratului umbră intersectează axa Ox în punctele *1*, *2*, *3* şi *4*.

Se unesc printr-o dreaptă punctele *4* şi m_1, dreaptă cu ajutorul căreia se obţine proiecţia k_1, adică :

$$k_1 = 4 - m_1 \cap kk_0.$$

În continuare se procedează în mod analog găsindu-se astfel proiecţia $m_1n_1p_1k_1$ a umbrei bazei *MNPK*.

Mărimea reală a acestei umbre se obţine rabătînd planul *P* pe planul *H* împreună cu umbra conţinută.

Este suficient, în acest sens, să se facă rabaterea unui singur vîrf, de exemplu k_1, care se aduce în planul *Q* (cu ajutorul unei orizontale a planului *P*), în poziţia *K*. Acest punct are după rabatere poziţia k_{p0}, ceea ce conduce, cu ajutorul unei translaţii, la obţinerea vîrfului umbră k_p.

Utilizîndu-se aceeaşi corespondenţă de afinitate se obţin şi celelalte vîrfuri umbră. De exemplu, umbra k_p este unită prin linii drepte cu punctele *4* şi *3*, ceea ce conduce la găsirea umbrelor M_p şi B_p. Aceste din urmă puncte se unesc prin drepte cu punctele *1* şi *2*, ceea ce are ca rezultat obţinerea vîrfului umbră N_p. Se unesc apoi prin linii drepte punctele M_p cu *L* şi B_p cu L_1, completîndu-se astfel umbra reală a prismei dată pe planul *P* al diedrului de referinţă considerat.

Utilizînd axa de afinitate $\Delta_1 = H_1 \cap P$ se observă că ea se intersectează cu *MN* (*mn*) în punctul α. Acest punct este coliniar cu m_1 şi n_1, ceea ce reflectă corespondenţa de afinitate dintre faţa *MNPK* şi umbra $m_1n_1p_1k_1$ din punctul *P*.

3. Umbra cilindrului de revoluţie

Cilindrul considerat are cercul director $\Omega(\omega)$ situat în planul reprezentării *H*, deci este vertical. Raza de lumină $R(r, R_0)$ este o dreaptă de poziţie generală şi s-a utilizat în epură, fig. 2, în scopul determinării umbrelor centrului $\Omega_1(\omega_1)$ al bazei superioare.

Umbra cercului Ω din planul *H* este însăşi acest cerc.

Se consideră prin $\Omega_1(\omega_1)$ raza de lumină $R(r)$. Intersecţia acesteia cu planele *H* şi *P* are ca rezultat obţinerea umbrelor Ω_h şi Ω_p. Pentru aceasta se utilizează planul $\Pi \perp H$ care trece prin raza *R*. Se rabate acest plan împreună cu raza de lumină şi cu intersecţia Π_p dintre planul *P* şi planul ajutător Π. Se observă că :

$$\Omega_h = R_0 \cap r \equiv \Pi_h \quad \text{şi} \quad \Omega_p = R_0 \cap \Pi_{1p},$$

deci :

$$\Omega_h = R \cap H, \qquad \Omega_p = R \cap P;$$

p este apoi proiectată ortogonal pe planul *H* al reprezentării. Prin urmare :

$$\Omega_p\, \omega_p \perp r \equiv \Pi_h \equiv \Pi_p.$$

Umbra cercului $\Omega_1(\omega_1, 6)$ este cercul Ω_h de aceeaşi rază cu Ω_1 pentru că acest cerc şi umbra sa sînt figuri plane congruente. Umbra aceluiaşi cerc Ω_1 pe planul P este elipsa Ω_p a cărei construcţie se dă în continuare. Se face mai întîi menţiunea că între cercul umbră şi elipsa umbră Ω_p se poate stabili o corespondenţă de afinitate a cărei axă este $Ox \equiv \Delta \equiv \delta = H \cap P$.

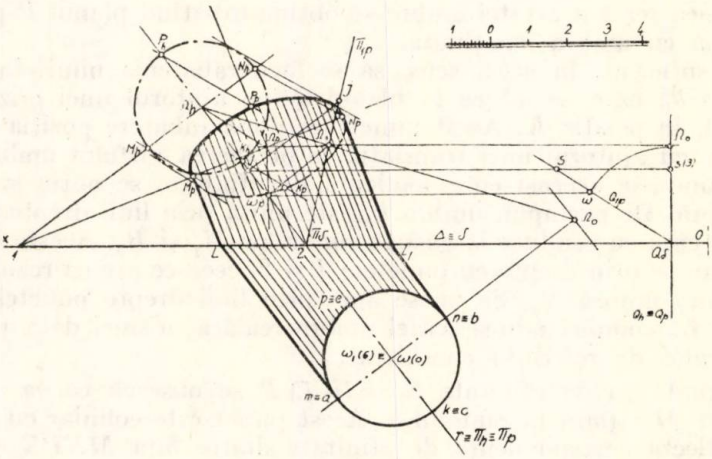

Fig. 2

De observat că urma Π_{1p} este determinată de punctele $\Pi_\delta = \Pi_h \cap Ox$ şi $j(j\,;\,z_j)$ în care $Z_j = Z_s$, ca puncte ce aparţin aceleiaşi orizontale a planului P. Se rabate şi centrul elipsei umbră pe planul H. În acest scop se consideră orizontala planului P ce trece prin acest punct. De aceea prin proiecţie Ω_p se duc paralele la axa Ox, ce intersectează urma Q_{1p} în Ω_p. Arcul de cerc al rabaterii acestui punct cu centrul în Q_δ şi de rază $Q_\delta\Omega_p$ intersectează urma $Q_h \equiv Q_p$ în punctul Ω_{0p}. Orizontala acestui punct, rabătută şi ea, intersectează perpendiculara la axa Ox dusă prin ω_p în Ω_{1p} care este centrul elipsei umbră. Perpendiculara $\Omega_{1p}\omega_p$ este urma planului proiectant faţă de $\Delta \equiv Ox$, în care se află traiectoria circulară a rabaterii centrului Ω_p

Pe baza corespondenţei indicate mai sus, se consideră diametrii principali în cercul umbră şi anume M_hN_h şi P_hK_h, care prelungiţi intersectează axa de afinitate $Ox \equiv \delta \equiv \Delta$ în punctele *1* şi *2*. Unindu-le prin drepte cu centrul umbră Ω_{1p} se obţin dreptele afine diametrelor principali indicaţi. Pe aceste linii afine se găsesc diametrii conjugaţi ai elipsei umbră din planul P. Proiectantele din extremităţile M_h, N_h, P_h şi K_h paralele cu dreapta $\Omega_h\,\Omega_{1p}$, intersectează dreptele afine $1 - \Omega_{1p}$ şi $2 - \Omega_{1p}$, în punctele M_p, N_p, P_p şi K_p, care sînt extremităţile diametrilor conjugaţi ai elipsei Ω_{1p} şi care permit trasarea acesteia. Umbra din planul orizontal a cilindrului considerat este dată de cercurile $\Omega(\omega)$ şi Ω_h completate cu tangentele exte-

rioare $A(a)M_h$ şi $B(b)N_h$. Porţiunea $LacbL_1$ este reală (văzută), iar porţiunea $LP_pN_pL_1$ este umbra reală din planul P a aceluiaşi corp, umbră care s-a rabătut pe planul H împreună cu planul P. Se observă că, de fapt, prin punctele de frîngere L şi L_1, umbra din planul P s-a completat cu tangentele la elipsa Ω_{1p}.

4. Umbra conului circular drept

Se consideră că baza conului (cercul director) este situată pe planul reprezentării H. Şi în acest caz planul de proiecţie p este determinat de axa $Ox \equiv \Delta \equiv \delta$ şi de un punct $M(m; 3)$ (fig. 3). Raza de lumină R este determinată de proiecţia sa r pe planul reprezentării H şi poziţia rabătută R_0, deci : $R(r; R_0)$.

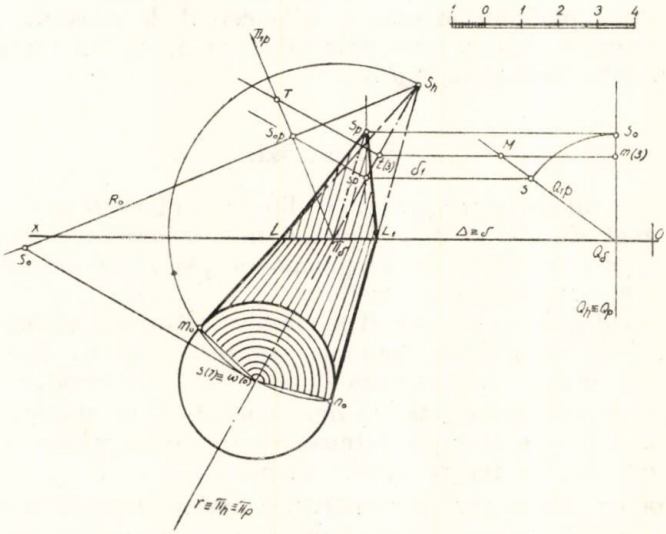

Fig. 3

Se duce prin R, ca dreaptă de poziţie generală, un plan vertical, ce corespunde notaţiei din epură, $r \equiv \Pi_h \equiv \Pi_p$; acest plan Π se intersectează cu planul P după dreapta $\Pi_p = \Pi \cap P$. Rabătînd planul Π pe H împreună cu elementele conţinute, se observă că dreptele R_0, Π_{1p} şi $r \equiv \Pi_h$ se intersectează astfel : $R_0 \cap r \equiv \Pi_h = S_h$, ceea ce corespunde intersecţiei $R \cap H = S_h$, sau $R_0 \cap \Pi_{1p} = S_p$, ceea ce corespunde intersecţiei $R \cap P = S_p$; punctele S_h şi S_p fiind umbrele pe cele două plane H şi P ale diedrului de referinţă, corespunzătoare vîrfului conului dat. Umbra pe planul H a acestui corp se completează ducînd din S_h tangentele S_hm şi S_hn la cercul Ω. Generatoarele $SM(sm)$ şi $SN(sn)$ separă suprafaţa laterală a conului în două

regiuni. Astfel, porţiunea $SMIN(smin)$ este luminată, iar porţiunea $SMJN$ $(smjn)$ este în umbră. Aceasta reprezintă ceea ce se numeşte umbra proprie a corpului geometric considerat. Umbra de pe planul H, umbra purtată sau aruncată, S_hmjn, prezintă şi ea două porţiuni şi anume umbra S_hLL_1 din H_p care este virtuală, şi porţiunea LL_1njm reală, ca fiind situată în semiplanul H_a.

Pentru a construi umbra aceluiaşi con pe planul P este suficient să se facă ridicarea planului Π împreună cu S_p şi apoi să se rabată planul P pe planul H împreună cu aceeaşi umbră. În acest scop se duce prin S_p perpendiculara la $r \equiv \Pi_h$, al cărei picior este proiecţia S_p. Se consideră orizontala planului P ce trece prin acest punct (proiecţia pe planul H), care intersectează urma Q_{1p} a planului proiectant ce conţine punctul M, în punctul S_p. Rabaterea planului P şi a punctului umbră se reprezintă în epură prin arcul de cerc sS_0 cu centrul în Q. Prin proiecţia s_p se duce o perpendiculară la axa Ox (urma planului în care se află cercul de rabatere al punctului S_1), care se intersectează cu orizontala rabătută Δ_{10} în S_p. Umbra pe planul P a conului este triunghiul S_pLL_1.

5. Concluzii

1. În sistemul de referinţă format din două plane H şi P, primul fiind planul reprezentării, iar al doilea planul de proiecţie, un corp geometric se reprezintă prin proiecţia sa ortogonală pe planul H la care se adaugă cotele diferitelor puncte caracteristice.

2. În cazul prismei drepte şi a cilindrului drept, umbra proprie nu este văzută, deoarece se proiectează pe conturul aparent din planul H al reprezentării şi anume în porţiunea de contact cu umbra aruncată.

3. Folosind corespondenţa de omologie afină se elimină multe operaţii de repetare, ca de exemplu intersecţia dreptei cu planul, ceea ce simplifică în mod sensibil reprezentările plane.

4. Metoda indicată pentru construcţia umbrelor poate fi utilă în proiectare în domeniul construcţiilor, arhitecturii, mecanicii etc.

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Institutul politehnic, Iaşi,
Catedra de desen şi geometrie descriptivă

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SCHATTEN DER KÖRPER DURCH AFFINER HOMOLOGIE-KORRESPONDENZ

(Zusammenfassung)

Es wird die Konstruktion der eigenen und getragenen Schatten der geometrischen Körper, auf die Ebenen des Referenzdieders behandelt.

Offensichtlich, ist ein Teil der getragenen Schatten reell, während ein anderer Teil virtuell ist.

Jedenfalls konnte die Konstruktion der getragenen Schatten in diesem allgemeineren Rahmen, mittels der affinen Korrespondenz, die sich zwischen den Elementen des gegebenen Körpers, durch seiner Projektion in der Ebene H der Darstellung, und die zu konstruierenden Schatten feststellt, realisiert werden.

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C.D. 537.52

LA CARACTÉRISTIQUE COURANT—TENSION À LA DÉCHARGE ÉLECTRIQUE SUR UN ÉLECTROLYTE [1])

PAR

N. CALINICENCO et N. A. MATEESCU

1. Introduction

On trace les caractéristiques courant-tension à la décharge électrique produite entre une pointe métallique et un électrolyte.

2. Méthode de mesure

La pointe était en cuivre et constituait la cathode, et l'électrolyte était de l'eau habituelle et constituait l'anode. La distance entre les électrodes était $d = 5$ mm.

3. Résultats et interprétation

Les caractéristiques ont été tracées pour différentes pressions (la pression était maintenue constante pour la même caractéristique) comprises entre 29 torr et 260 torr.

Le domaine dans lequel a été varié la tension de la source d'alimentation était contenu entre 600 V et 1 425 V. Dans le circuit a été introduit une résistance de 20 KΩ.

On observe que toutes ces caractéristiques ont au commencement une portion de proportionnalité entre la tension appliquée et le courant de décharge ; puis le courant de décharge atteint une valeur de saturation. Nous reproduisons ici quatre de ces caractéristiques (fig. 1...4).

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[1]) Présentée lors de la session scientifique de l'Institut Polytechnique de Jassy, le 2–4 juillet 1970.

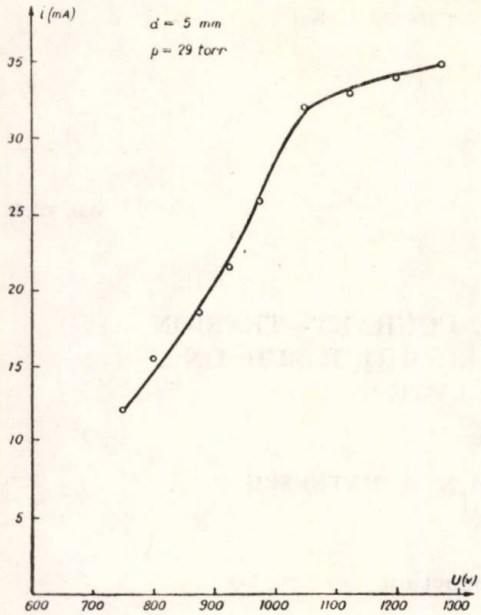

Fig. 1

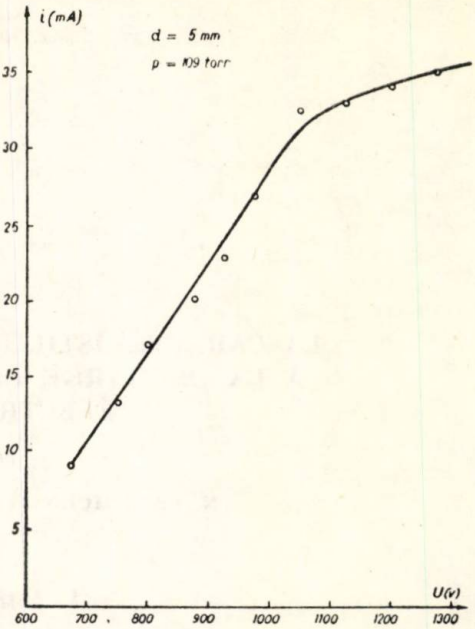

Fig. 2

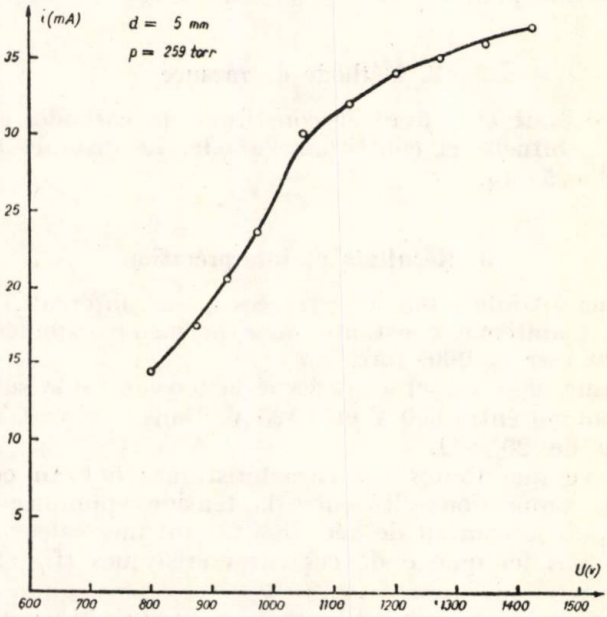

Fig. 3

La décharge étudiée est une décharge luminescente anormale.

Nous expliquons la saturation du courant de la décharge comme suit : on sait qu'à la décharge luminescente anormale les ions et les photons formés dans la lumière négative tombent sur la cathode et produissent une émission secondaire qui maintient la décharge.

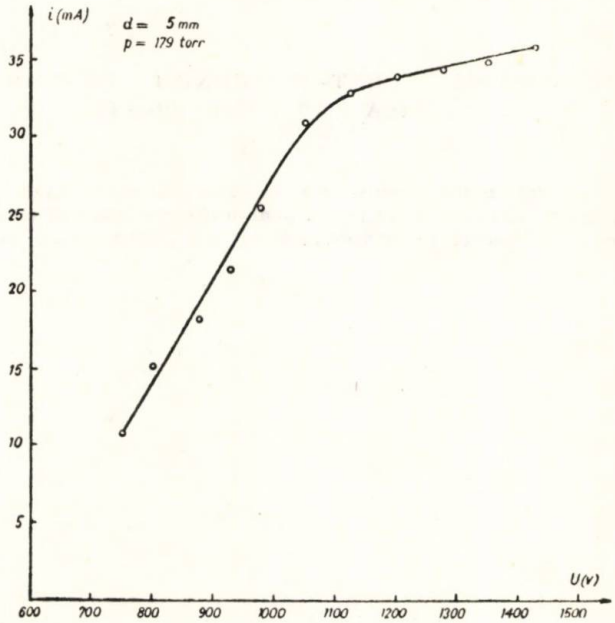

Fig. 4

Nous considérons qu'à cette décharge l'émission secondaire par cathode a un rôle important.

Jusqu'à la tension correspondante au courant de saturation le nombre des ions et des photons qui se forment croît avec la tension et par conséquent le courant de décharge croît. À partir de cette tension le nombre des ions et des photons reste constant, donc l'émission secondaire par cathode reste constante, ce qui fait que le courant de la décharge devient constant.

4. Conclusions

La décharge électrique produite entre une cathode métallique et une anode électrolyte ressemble à la décharge électrique produite entre des électrodes métalliques.

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Institut Polytechnique, Jassy,
Chaire de Physique

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CARACTERISTICA CURENT–TENSIUNE LA DESCĂRCAREA ELECTRICĂ PE UN ELECTROLIT

(Rezumat)

Se trasează caracteristica curent-tensiune la descărcarea electrică produsă între un vîrf metalic (care este catod) şi electrolit, pentru diferite presiuni între 29 şi 260 torr Se constată că această descărcare se aseamănă cu cea produsă între electrozi metalici

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C.D. 543.42; 661.83

INTENSIFICATION DES RAIES SPECTRALES SOUS L'INFLUENCE DES MÉTAUX ALCALINS [1])

PAR

GH. TACU et MARCEL MOISE

1. Introduction

L'accroissement de la sensibilité de l'analyse spectrale d'émission et par la suite le rabaissement des limites de décélement des éléments a préoccupé nombre de chercheurs [1]...[9]. L'une des méthodes employées couramment consiste dans l'introduction de certains sels des métaux alcalins dans les échantillons à analyser.

La présence des métaux alcalins détermine des phenomènes spécifiques dans le plasma de décharge, entraînant une certaine influence sur l'émission des atomes présents dans la source spectrale. Grâce à l'énergie d'ionisation réduite, les métaux alcalins font baisser la température de l'arc pendant la durée de leur volatilisation. En même temps il se produit une stabilisation de la décharge par l'action tampon de leur sels. L'ionisation partielle des atoms des métaux alcalins entraîne une augmentation du nombre des électrons dans le plasma et par la suite une plus grande probabilité d'excitation de certaines raies spectrales.

Suivant certains auteurs [6], l'influence des métaux alcalins sur l'intensité des lignes spectrales se manifeste pour les éléments ayant un potentiel d'excitation réduit. Par l'abaissement de la température de l'arc on empêche la formation d'oxydes à grande énergie de dissociation, qui diminuent le nombre des atomes neutres capables d'une émission de certaines raies spectrales.

Par exemple, l'alluminium, le calcium et en général les métaux alcalino-terreux donnent des oxydes réfractaires à la suite des réactions chimiques qui se produisent dans le plasma de la décharge.

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[1]) Présentée lors de la session scientifique de l'Institut Polytechnique de Jassy, le 2–4 juillet 1970.

On peut éviter la formation d'oxydes difficiles à séparer par la décharge dans une atmosphére privée d'oxygène (décharge dans une atmosphère de gaz nobles).

En cas d'emploi de la décharge entre les électrodes de carbone apparaisent les bandes du cyane. Une action importante des sels de métaux alcalins est de réduire beaucoup l'excitation des molécules de cyane par la réduction de la température du plasma au dessous d'une certaine valeur. On a ainsi la possibilité d'utiliser dans des bonnes conditions certaines lignes analytiques qui sont situées dans le domaine de ces bandes. L'excitation de la molécule de cyane est réduite pendant l'action tampon du métal alcalin correspondant.

Un phénomène particulier qui conduit à l'intensification de certaines lignes spectrales pour certains éléments, en présence d'atomes des métaux alcalins dans le plasma, est le transfert d'énergie qui peut avoir lieu à la suite des chocs de deuxième ordre entre les atomes de certains éléments et les atomes des métaux alcalins. Certains atomes possèdent dans des conditions spéciales des états d'excitation à partir desquels l'électron est incapable de revenir spontanément à son état normal.

En cet état métastable la durée de vie de l'état d'excitation est prolongée et la probabilité des chocs de deuxième ordre est augmentée.

Un atome qui se trouve dans un état métastable peut revenir à un état normal soit par un choc de deuxième ordre avec un autre atome, soit par excitation vers un état supérieur non métastable.

Si le niveau supérieur de l'atome excité d'un élément est similaire en ce qui concerne l'énergie au niveau possible d'un atome nonexcité d'un autre élément, il est plus probable un transfert d'énergie dû aux chocs de deuxième ordre. Le transfert d'énergie a lieu dans des conditions de résonance. Mais puisque les probabilités de transition des lignes sensibles intenses sont très grandes et par conséquant la periode d'existance de ces niveaux supérieurs est très petite, ces lignes sont peu affectées par les chocs de deuxième ordre et leurs niveaux supérieurs vont jouer probablement un rôle plus petit dans le transfert d'énergie.

Une théorie unitaire sur l'intensification des lignes spectrales sous l'action du troisième élément n'a pas encore été établie.

En fonction des caractéristiques de la décharge, des caractéristiques des éléments excitants dans le plasma de décharge, des influences réciproques entre les atomes des éléments analysés etc, un ou plusieurs des phénomènes qui conduisent à l'intensification des lignes spectrales est prépondérant.

Dans cette Note nous avons étudié les phénomènes d'intensification des lignes spectrales de certains éléments des minéraux, par l'introduction de sels de métaux alcalins dans les échantillons à analyser.

2. Appareillage employé et méthode de travail

On a utilisé un spectrographe du type FD-S à trois prismes en verre (Row-Rathenow RDG). Le spectrographe donne un spèctre compris entre 3 900 Å et 6 000 Å sur une plaque de 9×24 cm avec une dispersion qui varie entre 3,5 Å/mm pour une longueur d'onde de 3 870 Å et 20,4 Å/mm pour une longueur d'onde de 5 700 Å.

Comme source spectrale on a employé un arc à courant continu entre des éléctrodes de carbone. La tension d'alimentation de l'arc a été de 120 V, le courant de la décharge étant de 8 Å à une distance entre les électrodes de 4 mm. Les électrodes employés ont été le carbone spectral du type S.U d'un diamètre de 5 mm. L'anode a été percée à une profondeur de 2 mm, le diamètre du trou étant de 3,5 mm. La cathode a le diamètre de 5 mm avec la pointe tronconique à 45°. Comme recepteurs pour les radiations on a employé des plaques spectrales ORWO-Rot Rapid WP 1.

L'analyse spectrale qualitative a été réalisée avec un spectroprojecteur type SP2. Pour l'analyse quantitative on a utilisé un microphotomètre M.F2 Karl Zeiss Jena.

Le but du travail a été l'analyse de l'éffet de l'influence de quelques sels de métaux alcalins sur le degré de noircissement des lignes spectrales. Cette influence a été étudié dans le cas de deux minéraux différents (*A* et *B*) dont l'un était un concentré de plomb.

Le minerai *A* a été mélangé en proportion de 1%, 10% et 50% avec les sels NaCl, $CsNO_3$, Li_2CO_3, RbCl et $K_2Cr_2O_7$. Les échantillons ont été introduits par quantités de 25 mg dans l'orifice de l'éléctrode de carbone qui servait d'anode. La décharge de l'arc pour chaque échantillon analysé a été enregistrée pendant une durée de 60 s avec une fente du spectrographe de 10 μm et une fente intermediare de 1,2 mm. Le minerai *B* a été mélangé en proportion de 1%, 3,33% et 10% avec les sels NaCl, $CsNO_3$, RbCl, Li_2CO_3 et KCl. Les échantillons ont été introduits par quantités de 50 mg dans l'orifice de l'électrode de carbone qui servait d'anode. Dans le cas du minerai *B* la décharge de l'arc pour chaque échantillon analysé a été enregistré pendant 45 s avec une fente du spectrographe de 10 μm et avec une fente intermédiaire de 2 mm.

Les spèctres obtenus ont été analysés au microphotomètre à l'aide des lignes présentant un changement de noircissement, en prenant comme base de comparaison le noircissement du spèctre du même minerai sans addition de sels des métaux alcalins.

3. Résultats de la recherche

L'analyse qualitative des minerais *A* et *B* a indiqué les éléments suivants :

Minerai *A* : Al, Ba, Cd, Co, Cu, Fe, Mn, Na, Pb, Si, Tl, Zn, V.

Minerai *B* : Al, Ba, Ca, Cd, Cr, Cu, Fe, Ga, In, Mg, Mn, Na, Pb, Si, Te, Zn.

L'analyse qualitative des noircissements des lignes spectrales produits par l'action des sels des métaux alcalins a montré l'existence d'une influence particulière sur quelques éléments qui se trouvent dans les deux minerais. Ainsi dans le minerai *A* l'influence s'est manifestée pour Al, Ba et Tl et dans le minerai *B* pour Ba, Ca, Ga et In.

Du point de vue quantitatif l'influence des sels des métaux alcalins sur le noircissement des lignes spectrales de ces éléments a été déterminee par des mesures photométriques sur les lignes correspondantes.

Le noircissement des lignes a été rapporté au noircissement de la ligne de base (sous l'action du métal alcalin). On a déterminé le degré de noircissement

$$G = \frac{S_l}{S_b},$$

qui représente le rapport entre le noircissement de la ligne intensifiée (S_l) et le noircissement de la ligne de l'élément du minerai sans addition de sel (S_b).

En ordonnant les degrés de noircissement des lignes spectrales des éléments étudiés sous l'action de divers sels des métaux alcalins, en fonction de la différence entre l'énergie d'ionisation et l'énergie d'excitation de la ligne on a constaté une dépendance c'est-à-dire : pour une certaine concentration du sel du métal alcalin par exemple 3,33% ou 10% le degré de noircissement croît avec la diminution de la différence entre l'énergie d'ionisation et l'énergie d'excitation de la ligne.

Tableau 1

Raie spectrale		Noircissement					$E_i - E_{ex}$
		NaCl	$CsNO_3$	RbCl	Li_2CO_3	$K_2Cr_2O_7$	eV
Ba–II–4554	S_l	1,385	–	1,060	–	1,379	
	S_b	0,375	–	0,336	–	0,375	2,07
	G	3,69	–	3,15	–	3,67	
Tl–I–5350,5	S_l	0,667	0,524	0,434	0,284	0,475	
	S_b	0,226	0,200	0,183	0,183	0,226	2,84
	G	2,94	2,61	2,37	1,55	2,10	
Al–I–3961	S_l	1,850	1,399	0,981	–	2,106	
	S_b	1,197	0,676	0,626	–	1,197	2,84
	G	1,57	2,07	1,56	–	1,84	

Dans les tableau 1 on donne les degrés de noircissement de certaines lignes spectrales d'éléments contenus dans le minerai *A*, en fonction du sel du métal alcalin, pour une concentration de 10%, ordonné d'après l'accroissement de $E_i - E_{ex}$.

Tableau 2

Raie spectrale		Noircissement					$E_i - E_{ex}$
		NaCl	$CsNO_3$	RbCl	Li_2CO_3	KCl	eV
Ca – I – 4318,7	S_l	0,570	–	0,865	–	0,765	
	S_b	0,083	–	0,132	–	0,083	1,34
	G	6,87	–	6,55	–	9,22	
Ba – II – 4554	S_l	–	–	0,343	0,312	0,460	
	S_b	–	–	0,120	0,096	0,096	2,07
	G	–	–	2,86	3,25	4,79	
In – I – 4101,8	S_l	0,579	0,776	0,801	0,563	0,673	
	S_b	0,388	0,590	0,591	0,388	0,388	2,77
	G	1,50	1,31	1,35	1,45	1,73	
Ga – I – 4172,1	S_l	0,549	0,791	0,852	0,560	0,686	
	S_b	0,472	0,630	0,630	0,472	0,472	2,9
	G	1,16	1,26	1,35	1,18	1,46	

Le tableau 2 présente les valeurs similaires obtenues pour le minerai *B*.

Pour une concentration de 10% de RbCl et NaCl dans le minerai *B* apparait' une raie intense du strontium, à savoir la raie ultime Sr – 4 607,30Å. Dans l'analyse qualitative on n'a pas mis en évidence l'existence de cette ligne. Ceci montre le fait que dans le minerai *B* il y en a des traces de strontium, et que les sels des metaux alcalins produisent une intensification de la raie ultime.

Malgré le fait que les deux minerais contiennent du thalium on a constaté un accroissement du noircissement de la ligne Tl – 5 350 Å sous l'influence des sels des métaux alcalins utilisés, seulement pour le minérai *A*.

En effectuant l'analyse quantitative du thalium dans les deux minerais par la méthode des 3 étalons [10], [11] on a obtenu les resultats suivants. Dans le minerai *A* le thalium se trouve en concentration de $6{,}6.10^{-4}$ % et dans le minerai *B* en concentration de 6.10^{-3} %. Des résultats similaires ont été obtenus avec une concentration de 3,33% pour le minerai *B*.

Les résultats obtenus qui se réfèrent au thalium indiquent le fait que pour les concentrations du thalium plus grandes que 10^{-3} % on n'obtient pas d'intensifications de la ligne mentionnée plus haut.

4. Conclusions

1. Les sels des métaux alcalins utilisés produisent un accroissement de l'intensité des lignes spectrales pour les elements Al, Ba, Ca, Ga, In et Tl qui se trouvent dans une concentration très faible dans les minerais à analyser.

2. Pour les éléments analysés, on a constaté une dépendance entre le degré de noircissement d'une ligne et la différence entre l'énergie d'ionissation et l'énergie d'excitation de cette ligne sous l'action des sels des métaux alcalins ; on constate que plus cette difference est petite, plus est grand le degré de noircissement obtenu (par exemple dans le cas du Ca—4 319 Å, pour lequel $E_i - E_{ex} = 1{,}34$ eV, on obtient un degré de noircissement d'environ 9 (avec KCl 10%), pendant que pour Ga—4 172 où $E_i - E_{ex} = 2{,}90$ eV, on obtient un degré de noircissement d'environ 1,5).

3. Pour des petites valeurs de la difference $E_i - E_{ex}$ (au dessous de 2,1 eV) on constate une intensification aussi pour d'autres lignes, outres que les raies ultimes (c'est le cas par exemple pour le calcium et le barium).

4. Pour des grandes valeurs de $E_i - E_{ex}$ (au dessus de 2,7 eV) seulement les raies ultimes sont intensifiées et ce phénomène a lieu seulement pour des faibles concentrations (par exemple, dans le cas du thalium, l'intensification des lignes Tl—5 350 Å apparait seulement pour une concentration moindre que 10^{-3} %).

5. L'intensification des lignes spectrales sous l'action des sels des métaux alcalins conduit à un accroissement de la sensibilité et de la précision de l'analyse quantitative des traces des certains éléments dans les minéraux ; l'accroissement de la sensibilité d'analyse permet à déterminer des très petites concentrations de certains éléments dans les minerais.

6. L'augmentation de la précision de détermination est liée aussi au fait que par l'intensification des lignes spectrales faibles on peut atteindre le domaine des expositions normales.

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INTENSIFICAREA LINIILOR SPECTRALE SUB INFLUENŢA CÎTORVA SĂRURI ALE METALELOR ALCALINE

(Rezumat)

Se studiază influenţa sărurilor $K_2Cr_2O_7$, NaCl, $CsNO_3$, RbCl şi Li_2CO_3 asupra gradului de intensificare a unor linii spectrale. Se constată o mărire a intensităţii unor linii spectrale ale aluminiului, bariului, calciului, galiului, indiului şi taliului.

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MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

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ELEKTRETEN AUS CARNAUBAWACHS DIE UNTER DER SIMULTANEN EINWIRKUNG VON MAGNETISCHEN UND ELEKTRISCHEN FELDERN HERGESTELLT WURDEN [1])

VON

EUGEN OSADEŢ, C. PASNICU und EMIL LUCA

1. Einführungsbegrieffe

Neben den gewöhnlichen Bildungsvorgänge der Elektreten und zwar :

1° Erwärmung der Proben unter Einwirkung eines starken elektrischen Feldes (Thermoelektreten) [1]...[7] ;

2° Beleuchtung der Proben unter gleichzeitige Einwirkung des elektrischen Feldes (Photoelektreten) [8]...[11] ;

3° Einfache Einwirkung des elektrischen Feldes (Elektroelektreten) [12], [13] ;

4° Einwirkung der starken radioaktiven Strahlungen (Pseudoelektreten) [14], gelang es Khare und Bhatnagar [15]...[17] in der letzten Zeit, Elektreten unter der Einwirkung magnetischen Felder auf verschiedene Stoffe, herzustellen, die sogenannten Magnetoelektreten.

Die bildung der Thermoelektreten, mit Erscheinung der *Hetero* und *Homoladungen*, ist nach Gross [18] nicht nur eine Folge der Existenz von ionischen Ladungen in den Proben und von dipolaren Molekülen, die sich unter dem Einfluss des elektrischen Polarisationsfeldes orientieren können, sondern auch des Übergangs von elektrischen Ladungen zwischen Elektroden und Probe, durch die Durchbrechung des Zwischenraumes (Interstitiums) zwischen diesen.

Diese Theorie wurde durch die Versuche von Gemant [4], Gutmann [14] und anderen Autoren, bestätigt.

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[1]) **Mittgeteilt an der wissenschaftlichen Tagung der Technische Hochschule Jassy, am 3.–4. Juli 1970.**

Da die Bildung von Magnetoelektreten mit der Erscheinung von *Idioladungen* und *Isoladungen*, nach Khare und Bhatnagar [15]... [17], nicht durch den Übergang von elektrischen Ladungen zwischen Elektroden und Probe, infolge der Durchbrechung des Zwischenraumes, zustande kommen kann, wird sie, nach dennen, sowohl durch die Gefangenahme von Ionen in mikroskopischen Inhomogenitäten der Probe, die zur Erscheinung der Isoladungen führt, als auch durch die Existenz der dipolaren Molekülen, die die Idioladungen mit sich bringt, verursacht.

Jedoch fordert die Bildung eines Elektrets nicht nur die Anwesenheit von solchen ionischen Ladungen und dipolaren Molekülen, sondern auch die Orientierung dessen in einer bestimmten Richtung.

Wenn dieser Orientierungsvorgang, im Falle der im elektrischen Feld gebildeten Elektreten, einfach erscheint, so wird die Frage schwieriger, wenn es sich um die Magnetoelektreten handelt, alzo wenn von der Orientierung der dipolaren Molekülen im magnetischen Feld die Rede ist.

Nach McMahon [19] und Selwood [17], konnte die Orientierung der polaren Molekülen im magnetischen Feld als einen in Magnetfeldrichtung anisotropischen Wachsturmvorgang, also als ein Vorhandensein einer diamagnetischen Anisotropie, betrachtet Werden. Unter der Einwirkung des magnetischen Feldes versuchen diese Moleküle ihre Achse der grössten Polarisabilität in Magnetfeldrichtung zu orientieren.

Es scheint also, dass nach Khare und Bhatnagar [17], die Bildung von Magnetoelektreten von der Anwesenheit von polaren Molekülen mit diamagnetischen Anisotropie im betreffenden Stoff abhängt.

Da diese Annahmen den ganzen Bildungsmechanismus der Elektreten nicht genügend erklären können, da nur unter der Einwirkung des elektrischen (Thermoelektreten) oder magnetischen Feldes (Magnetoelektreten) gearbeitet wurde, untersuchen wir hier, zuerst nur experimentell, wie die gleichzeitige Einwirkung von einem elektrischen und magnetischen Feld die Elektretsbildung beeinflusst.

Um unsere Ergebnisse mit denen aus der Literatur vergleichen zu können haben wir als Elektretsstoff Carnaubawachs verwendet, welches von Eguchi [1] bei der Herstellung des ersten Elektrets und von Bhatnagar [15] zur Herstellung des ersten Magnetoelektrets benutzt wurde.

2. Experimenteller Teil

Die parallelipipedischen Carnaubawachsproben (14 ×14 ×5 mm) wurden durch Pressen bei 300 at in einer metallischen Matrize aus einer bestimmten pulverformige Carnaubawachsmenge hergestellt.

Diese Proben wurden in einem kleinen Plexiglasskasten mit entsprechenden Abmessungen, zwischen zwei Kupferelektroden untergebracht. Die elektroden wurden an die Hochspannung-Gleichstromquelle geschaltet. Gleichzeitig sorgten die Kupferelektroden auch für den thermischen Kontakt zwischen der Probe und den Wänden des Kastens.

Der Kasten mit der Probe wurde wieder in einem Plexiglassgefäss gelegt, welches Öl mit regelbarer Temperatur enthielt, und sich zwischen den Polschuhen eines starken Elektromagnetes befand, so dass die Flachseiten der Carnaubawachsproben genau parallel zu den Polschuhen des Elektromagnetes und in der Richtung ihrer Zentren lagen.

Es gab auch die Möglichkeit, den Kasten mit der Probe um eine vertikale Achse zu drehen.

Für jede Messungsreihe wurden Serien von 5 identischen Proben hergestellt, von denen eine Probe als Vergleichsprobe diente, da sie nicht erwärmt wurde und sich nicht unter der Einwirkung der magnetischen und elektrischen Feldern befand. Alle andere Proben wurden bis 75°C erwärmt (etwa unter dem Schmelzpunkz des Carnaubawachses) und bei dieser konstanten Temperatur wurden sie eine Stunde lang der Einwirkung der elektrischen und magnetischen Felder unterworfen und zwar unter folgenden Bedingungen :

I-Serie : nur unter Einwirkung des elektrischen Feldes ;

II-Serie : nur unter Einwirkung des magnetischen Feldes ;

III-Serie : unter gleichzeitiger Einwirkung der elektrischen und magnetischen Felder in der gleichen Richtung, mit der negativen Polung des elektrischen Feldes an den Nordpol des Elektromagnetes ;

IV-Serie : unter gleichzeitiger Einwirkung der elektrischen und magnetischen Felder in der gleichen Richtung, mit der pozitiven Polung des elektromagnetischen Feldes an den Nordpol des Elektromagnetes ;

V-Serie : unter der simultanen Einwirkung der elektrischen und magnetischen Felder, die nun senkrecht aufeinander standen.

Bei allen Messungen war die elektrische Feldstärke $E = 6$ KV/cm und die magnetische Induktion $B = 2{,}2$ T.

Nach diesem Behandlungsvorgang wurden die Proben eine Stunde lang weiter unter der Einwirkung des elektrischen und magnetischen Feldes gelassen, bis sie die Raumtemperatur einnahmen.

Nun wurden die elektrischen und magnetischen Felder ausgeschaltet und es wurde die elektrische Ladung auf den Seitenflächen des so gebildeten Elektrets gemessen. Die Messungen wurden nach 24 Stunden wiederholt.

Bei der Messung der elektrischen Ladungen benüzten wir die bekannte Methode der elektrostatischen Induktion, mit Eichkondensatoren mit bekanter Kapazität und ein Elektrometer Orion TR 1051 dessen innerer Widerstand grösser als 10^{14} Ohm war und welches die elektrische Ladung der Grosse 10^{-12} C mit hinreichender Genauigkeit zu messen gestattete.

3. Ergebnisse

Die experimentellen Werte der elektrischen Ladung von den Elektretsoberflächen, die bei der Ausschaltung des elektrischen und magnetischen Feldes gemessen wurden und auch die zeitliche Änderung dieser Werte sind für alle Arbeitsbedigungen durch die Kurven im Abb. 1...5 dargestellt.

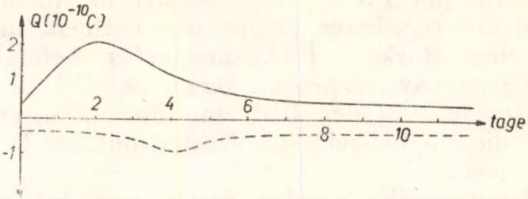

Abb. 1. — Der Ausgangswert und die zeitliche Änderung der elektrischen Ladung auf den Oberflächen des nur unter der Einwirkung des elektrischen Feldes von 6 KV/cm gebildeten Elektrets.

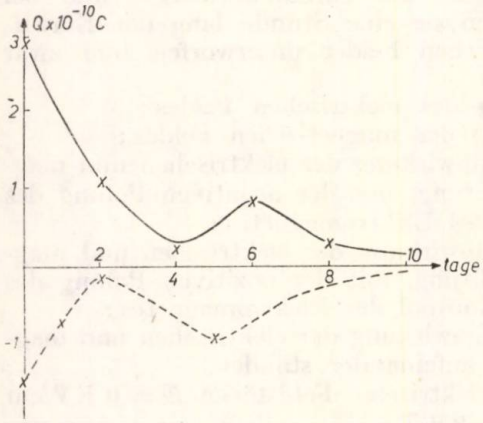

Abb. 2. — Der Ausgangswert und die Änderung der elektrischen Ladung auf den Oberflächen des nur unter der Einwirkund der magnetischen Induktion von 2,2 T gebildeten Elektrets.

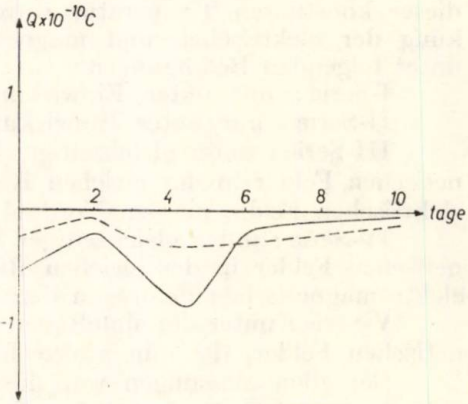

Abb. 3. — Der Ausgangswert und die zeitliche Änderung der Ladung auf der Oberfläche des unter gleichzeitiger Einwirkung des magnetischen und elektrischen Feldes mit paralleler Richtung gebildeten Elektrets, wobei die negative Polung des elektrischen Feldes an das Elektrod von der Seite des Nordpoles des Elektromagnetes geschaltet wurde.

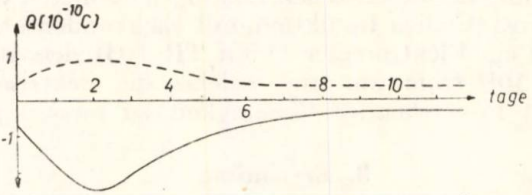

Abb. 4. — Der Ausgangswert und die zeitliche Änderung der Ladung auf der Oberfläche des unter gleichzeitige Einwirkung des magnetischen und elektrischen Feldes mit paralleler Richtung gebildeten Elektrets, wobei die pozitive Polung des elektrischen Feldes am Nordpol des Elektromagnetes geschaltet wurde.

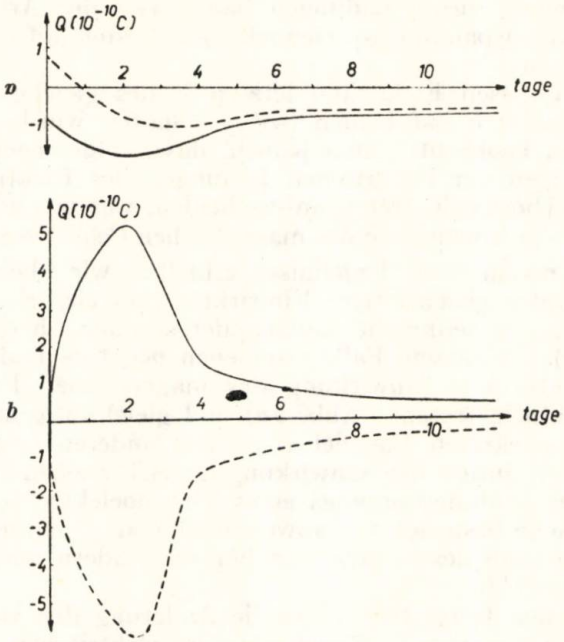

Abb. 5. – Der Ausgangswert und die zeitliche Änderung der Ladung auf der Oberfläche des unter gleichzeitiger Einwirkung des magnetischen und elektrischen Feldes mit senkrechter Stellung gebildeten Elektrets.

4. Diskussion

1. Die Kurven im Abb. 1 und 2, die den Ausgangswert und die zeitliche Änderung der elektrischen Ladung der durch Erwärmung im elektrischen Feld (Thermoelektret) und im magnetischen Feld (Magnetoelektret) gebildeten Elektreten darstellen, stimmen mit den Ergebnissen der Fachliteratur überein ; so sind die Ladungen auf den beiden Seiten des Thermoelektrets gleich und von umgekertem Vorzeichen während man bei dem Magnetoelektret am Ende die gleiche negative Ladung auf den beiden Seiten des Elektrets erhielt (K h a r e und B h a t n a g a r [16], [17]).

2. Die Kurven im Abb. 3 und 4 stellen den Anfangswert und die zeitliche Änderung der Ladung des unter der gleichzeitigen Einwirkung der parallelen elektrischen und magnetischen Felder gebildeten Elektrets dar und zeigen die Anwesenheit der elektrischen Ladungen von umgekerten Vorzeichen an den beiden Seiten des Elektrets, wie bei den Thermoelektreten,

Das Vorzeichen dieser Ladungen hängt von der Art der Shaltung der Polen der Hochspannungsspeisequelle im Bezug auf die Polung des Elektromagnetes ab.

Obwohl bei diesen Elektreten letzten Endes gleiche Ladungen mit Umgekerten Vorzeichen auf beiden Seiten erhalten wurden, wie bei den Thermoelektreten, beobachtet man jedoch, dass, sofort nach der Bildung, sich die Änderungen der elektrischen Ladungen des Elektretes, von diejenigen von den Thermoelektreten unterscheiden, was uns zu der Annahme führt, dass dafür die Einwirkung des magnetischen Feldes verantwortlich ist.

3. Die interessantesten Ergebnisse erhielten wir aber im Falle der Elektreten, die unter gleichzeitiger Einwirkung des elektrischen und magnetischen Feldes, die senkrecht aufeinander standen, hergestellt wurden (Abb. 5 *a* und *b*). In diesem Falle erschienen negative Ladungen auf den beiden Seiten unter der Einwirkung des magnetischen Feldes (entsprechend den Magnetoelektreten — Abb. 5*a*) und gleichzeitig erschienen auch Ladungen von umgekerten Vorzeichen an den anderen beiden Seiten des Elektrets, die sich unter der einwirkung des elektrischen Feldes befanden (entsprechend des Bildungsvorgangs eines Thermoelektrets — Abb. 5 *b*).

Diese Tatsache bestätigt die Anwesenheit von Orientierungsvorgänge nicht nur in Richtung des elektrischen Feldes, sondern auch in Richtung des magnetischen Feldes.

Es wurde auch festgestellt, dass die Änderung der Werte der elektrischen Ladungen (die auf die Einwirkung des elektrischen Feldes zurückzuführen sind), sofort nach dem Bildungsvorgang gemessen (also bevor ein stabiler Zustand erreicht wurde) und im Endzustand, grösser ist als die Änderung bei den Elektroelektreten, bei denen die elektrischen und magnetischen Felder gleichzeitig in derselben Richtung einwirkten oder nur eines von dieser Felder einwirkte.

Diese ersten experimentellen Ergebnisse gestatten uns zu behaupten, dass im Falle der gleichzeitigen Einwirkung des elektrischen und magnetischen Feldes, im Polarisationsvorgang der Verschiebung der Ionen und bzw. der Orientierung der Dipolen eines Elektretsstoffes solche Vorgänge stattfinden, für die eine Erklärung noch nöttig ist.

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ELECTREŢI DIN CEARĂ DE CARNAUBA FORMAŢI SUB ACŢIUNEA SIMULTANĂ A CÎMPURILOR ELECTRIC ŞI MAGNETIC

(Rezumat)

Se studiază experimental procesul de formare a electreţilor de ceară de Carnauba sub acţiunea simultană a unui cîmp electric şi a unuia magnetic.

Valorile sarcinilor electrice de pe feţele electreţilor astfel formaţi, măsurate imediat după formarea lor, cît şi variaţia lor în timp, denotă că în procesul de polarizare, respectiv de deplasare a ionilor şi de orientare a dipolilor substanţei electretice, intervin atît cîmpul electric cît şi cel magnetic, şi anume funcţie de orientarea liniilor de forţă ale cestor cîmpuri, paralel, antiparalel sau perpendicular.

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C.D. 537.31

CONTRIBUTIONS À L'ÉTUDE DES PROPRIÉTÉS ÉLECTRIQUES DE CERTAINES COUCHES MINCES DE SULFURE D'ARGENT (Ag_2S) DÉPOSÉES PAR VOIE CHIMIQUE [1])

PAR

VLAD CEHAN, PAMFILIA PETRARIU et EMIL LUCA

1. Considérations générales

Bien que les propriétés électriques des couches minces de Ag_2S ait constitué l'objet de nombreuses recherches ([1]...[3]), il reste pourtant beaucoup de problèmes, telle que la variation de la résistivité avec la fréquence, qui n'ont pas trouvé une solution complète.

C'est justement ce que nous a déterminé à effectuer des recherches sur la résistivité des couches minces de Ag_2S déposées chimiquement et à étudier leur variation avec la température et la fréquence.

2. Résultats expérimentaux et discussions

On a utilisé des couches minces de Ag_2S déposées chimiquement, par réduction avec glucose de la solution de $AgNO_3$ j'usqu'à Ag métallique, suivie du traitement des pellicules obtenues avec H_2S. On a employé comme support de petites plaques en verre.

L'épaisseur des couches minces, comprise entre 0,02 μ et 0,2 μ a été déterminé en pesant les petites plaques support avant et après la réalisation du dépôt. À cause de la non uniformité des couches déposées, les valeurs déterminées représentent les valeurs moyennes de l'épaisseur.

La mesure de la résistance électrique en fonction de la fréquence a été réalisée avec un Q-mètre TESLA BM-3 113, dans la gamme 5...50 kHz, à la température du milieu ambiant (23°C).

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[1]) Présentée lors de la session scientifique de l'Institut Polytechnique de Jassy, le 2–4 juillet 1970.

En vue de mesurer la variation de la résistance avec la température on a fait chauffer les preuves dans un four électrique, de 23°C à 100°C. Les mesures ont été effectuées en courant continu, à l'aide d'un galvanomètre TESLA DG-20.

Vue que les couches de Ag_2S présentent un puissant effet photorésistif, toutes les mesures ont été effectuées en maintenant les plaques dans l'obscurité.

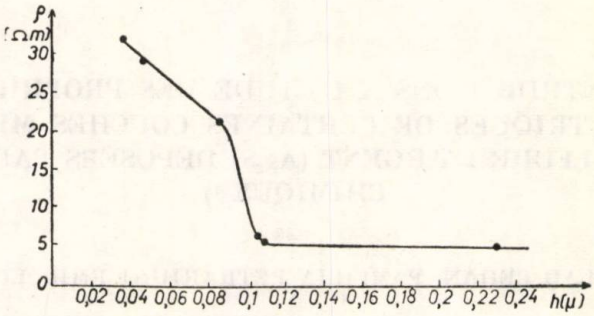

Fig. 1. — Variation de la résistivité en fonction de l'épaisseur de la pellicule.

La résistivité a été calculé en utilisant les valeurs de la résistance des pellicules, déterminées expérimentalement, et les dimensions géométriques.

On a représenté (fig. 1) la courbe de variation de la résistivité ρ avec l'épaisseur h de la pellicule de Ag_2S.

L'analise de la courbe obtenue relève un abaissement accentué de la résistivité jusqu'à des épaisseurs d'environ 0,09 μ, où un abaissement brusque intervient, qui se poursuit après plus lentement, lorsque l'épaisseur de la couche mince augmente.

Nous considérons que cette variation de la résistivité avec l'épaisseur est déterminée par le fait que les pellicules très minces de Ag_2S n'ont pas une structure continue, mais fragmentaire, de domaines. Jusqu'à une certaine épaisseur l'abaissement de la résistivité se produit surtout sur le compte de la fusion de pareils domaines, plus petites, pour en former d'autres, plus grandes, après quoi, aux épaisseurs plus grandes, l'abaissement de la résistivité a lieu sur le compte de l'épaississement des couches déposées.

La variation de la résistivité avec la température est présentée dans la fig. 2.

Il résulte de l'analyse des courbes obtenues que :

1° La variation avec la température est caractéristique aux semiconducteurs ; on constate un abaissement linéaire de la résistivité en fonction de $\ln t$ jusqu'aux environs de la température de 75°C, après quoi l'abaissement est plus lent, avec tendence vers des valeurs limites approchées pour toutes les épaisseurs.

2° Les couches plus minces ($h < 0{,}09\ \mu$) ont la pente des courbes $\rho = F(\ln t)$ plus grande que celles avec des épaisseurs plus grandes ($>0{,}09\ \mu$). On considère que la variation de la pente des courbes $\rho = F(\ln t)$ est déterminée par la différence de structure entre les pellicules à $h < 0{,}09\ \mu$ et celles à $h > 0{,}09\ \mu$, qu'on a décrite ci-dessus.

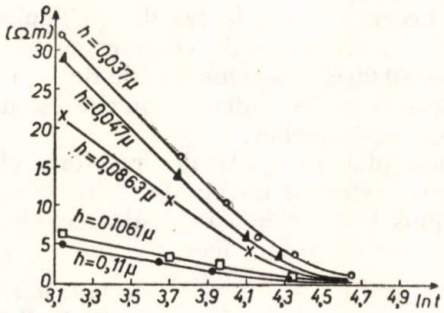

Fig. 2. — Variation de la résistivité en fonction de la température.

Dans le cas des couches plus minces, à structure fragmentaire, au fur et à mesure que la température augmente, l'agitation thermique des ions s'accroît, ce qui conduit à la réorganisation de leur structure, dans le sens d'une fusion des domaines; c'est ce que détermine l'abaissement

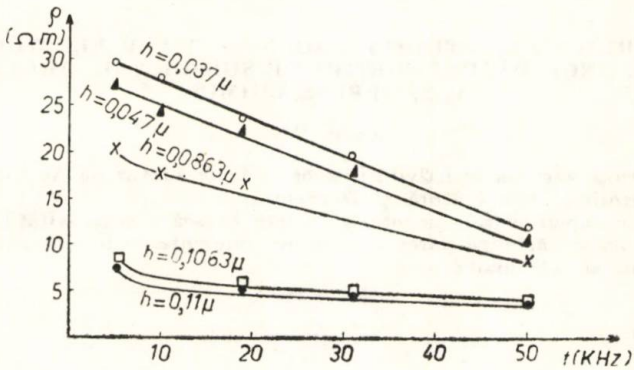

Fig. 3. — Variation de la résistivité en fonction de la fréquence.

de la résistivité. Pour les températures supérieures à 75°C les régions sont pour la plupart unies les unes aux autres, l'abaissement se produisant sur le compte de la croissance du nombres de porteurs. Les pellicules plus épaisses ayant une structure bien plus continue, l'abaissement, lors de l'aug-

mentation de la température, se produit surtout sur le compte de l'accroissement du nombre des porteurs.

La variation de la résistivité en fonction de la fréqence, présentée dans la fig. 3, indique un abaissement de ρ lors de l'augmentation de la fréquence.

On constate un abaissement de la résistivité lors de l'augmentation de la fréquence, plus accentué dans le cas des pellicules plus minces ($h < 0{,}09\ \mu$). Il existe une différence de comportement entre les pellicules à $h < 0{,}09\ \mu$ et celle à $h > 0{,}09\ \mu$, analogue à celle de la variation de la résistivité avec la température; nous considérons qu'elle est due toujours aux différences de la structure des couches.

Une interprétation plus complète des résultats obtenus sera donnée plus tard, lorsqu'on aura effectué un nombre plus grand de mesures, concernant un domaine plus large d'épaisseurs des couches minces, de même qu'une gamme plus vaste de fréquences.

Reçue le 1 novembre 1971

Institut Polytechnique, Jassy,
Chaire de Physique et d'Électronique

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CONTRIBUŢII LA STUDIUL PROPRIETĂŢILOR ELECTRICE ALE UNOR PĂTURI SUBŢIRI DE SULFURĂ DE ARGINT (Ag_2S) DEPUSE CHIMIC

(Rezumat)

Se cercetează variaţia rezistivităţii unor pelicule subţiri de Ag_2S depuse chimic în funcţie de grosime, temperatură şi frecvenţă.

Rezultatele experimentale denotă o scădere bruscă a rezistivităţii în jurul grosimii de 0,09 μ, datorată, după părerea autorilor, diferenţelor de structură dintre straturile mai subţiri şi cele mai groase.

Nr. coli tipar 11

Tiparul executat la Întreprinderea poligrafică Iaşi, str. Vasile Alecsandri nr. 12, sub cd. 36.
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MECANICĂ TEORETICĂ
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SECŢIA I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

SUMAR

BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
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Secţia I
MATEMATICĂ.
MECANICĂ TEORETICĂ. FIZICĂ

C.Z. 517.3

O PREZENTARE A INTEGRALEI RIEMANN PENTRU FUNCŢIILE DE O VARIABILĂ ŞI EXTENSIUNEA PROCEDEULUI LA UNELE FUNCŢIONALE DEFINITE ANALOG

DE

AL. CLIMESCU

1. Expunerea standard a integralei Riemann, aşa cum se poate găsi în indiferent care tratat contemporan de analiză clasică, prezintă numeroase avantaje printre care şi acela că se pretează la generalizări netriviale, posibilitate care a şi fost exploatată, în schimb însăşi definiţia integralei apare ca insuficient motivată. Din această cauză ne propunem să prezentăm aici integrala Riemann pe o cale mai scurtă şi mai elementară — deci, credem mai bine motivată — elementul fundamental şi posibil nou al demonstraţiei fiind compararea sumelor superioare (inferioare) Darboux pentru două partiţii ale intervalului care se găsesc într-o relaţie pe care o vom defini mai încolo. Procedeul se extinde aproape cuvînt cu cuvînt pentru alte funcţionale definite analog cu integrala, în general neliniare, spre deosebire de generalizările la care s-a făcut aluzie mai sus, care păstrează liniaritatea.

2. Notaţiile vor fi următoarele: partiţiile se notează prin litere mari P, Q,..., intervalul prin $[a, b]$, punctele de subdiviziune ale unei partiţii prin $a = x_0 < x_1 < \ldots < x_n = b$, $\min(x_i - x_{i-1})$ prin $d(P)$, $\max(x_i - x_{i-1})$ prin $\delta(P)$ (norma partiţiei P);

$$\sup f(x) = M, \quad \inf f(x) = m, \quad \sup f(x) = M_i, \quad \inf f(x) = m_i$$
$$[\text{pe } [a, b], \text{ respectiv pe } [x_{i-1}, x_i]];$$

integrala superioară respectiv inferioară prin

$$\mathrm{s}\int_a^b f(x)\mathrm{d}x, \qquad \mathrm{i}\int_a^b f(x)\mathrm{d}x;$$

în fine $S(P)$, $s(P)$ vor nota cele două sume Darboux.

$f(x)$ va fi o funcţie univocă oarecare definită şi mărginită pe $[a, b]$. Presupunem de asemenea cunoscute proprietăţile imediate ale sumelor D a r b o u x. Între partiţii introducem relaţia binară ρ prin definiţia

$$Q \rho P \Longleftrightarrow \delta(Q) \leqslant d(P).$$

1. *Dacă P are k puncte de subdiviziune interioare intervalului,*

$$Q \rho P \Rightarrow S(Q) \leqslant S(P) + k(M - m)\delta(Q).$$

Potrivit ipotezei, primul punct de subdiviziune din P, notat x_1, este fie şi punct de subdiviziune pentru Q, fie conţinut între două puncte consecutive din Q. În prima ipoteză porţiunea din $S(Q)$ referitoare la $[x_0, x_1]$ este cel mult egală cu $M_1(x_1 - x_0)$ şi putem continua compararea sumelor începînd cu $[x_1, x_2]$. În cazul al doilea fie

$$x_0 = y_0 < y_1 < \dots < y_p < x_1 < y_{p+1}, \qquad (y_{p+1} < x_2),$$

(y_i punctele de subdiviziune pentru Q).

Descompunem

$$M'_{p+1}(y_{p+1} - y_p) = M'_{p+1}[(x_1 - y_p) + (y_{p+1} - x_1)]$$

şi

$$M'_{p+1}(x_1 - y_p) = [M_1 + (M'_{p+1} - M_1)]\,(x_1 - y_p) = M_1(x_1 - y_p) +$$
$$+ (M'_{p+1} - M_1)(x_1 - y_p) \leqslant M_1(x_1 - y_p) + (M - m)\,(x_1 - y_p),$$

de unde

$$M'_1(y_1 - y_0) + \dots + M'_{p+1}(x_1 - y_p) \leqslant M_1(x_1 - y_p).$$

Termenul $M'_{p+1}(y_{p+1} - x_1)$ îl ataşăm porţiunii din $S(Q)$ referitoare la $[x_1, x_2]$ şi contribuţia lui la majorarea lui $S(Q)$ va fi $\leqslant (M - m)\,(y_{p+1} - x_1)$, de unde rezultă că contribuţia totală a intervalului $[y_p, y_{p+1}]$ va fi cel mult egală cu $(M - m)\,(y_{p+1} - y_p) \leqslant (M - m)\,\delta(Q)$.

Repetînd procesul de majorare pentru celelalte subintervale din P, ajungem în final, prin adunare, la inegalitatea din enunţ.

Similar se obţine

I'. $Q \rho P \Rightarrow s(Q) \geqslant s(P) - k(M - m)\delta(Q)$.

I şi I' au drept consecinţe uşor deduse principalele proprietăţi ale integralei şi anume :

$$\text{II.} \quad (\forall \varepsilon)\,(\exists \eta)\,\delta(Q) < \eta \Rightarrow s\int_a^b f(x)\mathrm{d}x \leqslant S(Q) < s\int_a^b f(x)\mathrm{d}x + \varepsilon$$

şi similar

$$i\int_a^b f(x)\mathrm{d}x - \varepsilon < s(Q) \leqslant i\int_a^b f(x)\mathrm{d}x.$$

Este suficient să se observe că

$$(\exists P)\ \mathrm{s}\int_a^b f(x)\mathrm{d}x \leqslant \mathrm{S}(P) < \mathrm{s}\int_a^b f(x)\mathrm{d}x + \frac{\varepsilon}{2}$$

și să se aplice I luînd

$$\eta \leqslant \frac{\varepsilon}{2k(M-m)}.$$

III. *Dacă P_n este un șir admisibil de partiții, $\delta(P_n) \to 0$ atunci*

$$\lim \mathrm{S}(P_n) = \mathrm{s}\int_a^b f(x)\mathrm{d}x, \quad \lim \mathrm{s}(P_n) = \mathrm{i}\int_a^b f(x)\mathrm{d}x.$$

Se observă că pentru $n > N$, $\delta(P_n) < \eta$ și se aplică II.

VI.
$$\mathrm{s}\int_a^c f(x)\,\mathrm{d}x + \mathrm{s}\int_c^b f(x)\,\mathrm{d}x = \mathrm{s}\int_a^b f(x)\mathrm{d}x,$$

$$\mathrm{i}\int_a^c f(x)\mathrm{d}x + \mathrm{i}\int_c^b f(x)\mathrm{d}x = \mathrm{i}\int_a^b f(x)\mathrm{d}x.$$

Este suficient a lua în III c ca punct de subdiviziune pentru fiecare P_n și a descompune $\mathrm{S}(P_n)$ în două sume parțiale corespunzătoare celor două intervale.

În cazul integrabilității rezultă de asemenea imediat criteriul lui D a r b o u x și formulele

$$\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x,$$

de unde se deduce integrabilitatea pe orice subinterval.

3. Demonstrațiile de mai sus se extind aproape cuvînt cu cuvînt la alte funcționale, construite analog cu integrala dar distincte de ea.

Luînd cel mai simplu exemplu fie $\varphi(x)$ o funcție continuă și crescătoare, $\varphi(-\infty) = 0$, $\varphi(\infty) = 1$.

Notăm

$$\mathrm{S}(P) = \Sigma\varphi(M_i)\,(x_i - x_{i-1}), \qquad \mathrm{s}(P) = \Sigma\varphi(m_i)\,(x_i - x_{i-1})\,;$$

cele două sume au următoarele proprietăți :

V. $0 \leqslant s(P) \leqslant S(P) \leqslant b - a$.

VI. Prin rafinare $s(P)$ creşte şi $S(P)$ descreşte.

VII. $(\forall P, \forall Q)\, s(Q) \leqslant S(P)$.

VIII. $Q \rho P \Rightarrow S(Q) \leqslant \delta(P) + k\delta(Q), \quad s(Q) \geqslant s(P) - k\delta(Q)$,

de unde cu denumiri şi notaţii analoge

$$\text{IX.} \qquad \varphi s \int_a^b f(x)\mathrm{d}x \leqslant S(Q) < \varphi s \int_a^b f(x)\mathrm{d}x + \varepsilon,$$

$$\varphi i \int_a^b f(x)\mathrm{d}x - \varepsilon < s(Q) \leqslant \varphi i \int_a^b f(x)\mathrm{d}x,$$

dacă $\delta(Q)$ este suficient de mic

III şi IV se extind la fel, adică

$$\text{X.} \quad \delta(P_n) \to 0 \Rightarrow \lim S(P_n) - \varphi s \int_a^b f(x)\,\mathrm{d}x, \quad \lim s(P_n) = \varphi i \int_a^b f(x)\mathrm{d}x,$$

$$\text{XI.} \qquad \varphi s \int_a^c + \varphi s \int_c^b = \varphi s \int_a^b, \quad \varphi i \int_a^c + \varphi i \int_c^b = \varphi i \int_a^b.$$

$\varphi i = \varphi s$ va defini de asemenea „integrabilitatea" în noul sens, cu consecinţe analoge acelora de la integrala R.

4. φs, φi există pentru cele mai generale funcţii posibile, de exemplu acelea care iau efectiv valorile ∞ sau $-\infty$ şi de asemenea se adaptează uşor pentru funcţiile multivoce.

În unele cazuri φ-integrala coincide cu integrala unei funcţii convenabil alese. În general însă φ-integrala este distinctă atît de integrala Riemann, cît şi de integralele Lebesgue sau Stieltjès, după cum ne putem convinge uşor luînd $f(x)$ egală pe [0, 1] cu funcţia lui Dirichlet, φ-integrabilă de valoare 0 dacă $\varphi(x)$ este constantă pe acest interval şi neintegrabilă altfel.

Primită la 4 decembrie 1972

Institutul politehnic, Iaşi,
Catedra de matematici I

UNE PRÉSENTATION DE L'INTÉGRALE DE RIEMANN POUR LES FONCTIONS D'UNE VARIABLE ET L'EXTENSION DU PROCÉDÉ À CERTAINES FONCTIONNELLES DÉFINIES D'UNE MANIÈRE ANALOGUE

(Résumé)

Un Lemme combinatoire (I, I') permet de démontrer d'une manière immédiate les propriétés fondamentales des intégrales, supérieure et inférieure d'une fonction bornée.

La méthode s'applique sans aucun changement important pour la φ - intégrabilité, où $\varphi(x)$ est continue et croissante, $\varphi(-\infty) = 0$,

$$\varphi(\infty)=1 \quad \text{et} \quad \varphi s \int_a^b f(x)\mathrm{d}x = \inf S(P), \qquad \varphi i \int_a^b f(x)\mathrm{d}x = \sup s(P),$$

et

$$S(P) = \Sigma\varphi(M_i)\,(x_i - x_{i-1}), \qquad s(P) = \Sigma\varphi(m_i)\,(x_i - x_{i-1}).$$

φs et φi existent pour n'importe quelle fonction même multivoque ou prenant la valeur $\pm\infty$; quelquefois la fonctionnelle φ - intégrale se réduit a l'intégrale Riemann mais en général elle est différente des intégrales connues.

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MECANICĂ TEORETICĂ. FIZICĂ

C.Z. 513.83

COFIBRĂRI CU SPAŢIILE BAZĂ ŞI TOTAL HAUSDORFF LOCAL COMPACTE [1])

DE

I. POP

1. În această Notă se defineşte coechivalenţa homotopică a două cofibrări (definiţia 1) şi se demonstrează teorema 1 care arată că există un functor covariant L' de la categoria ($\overset{h}{\mathrm{HLC}}$) (categoria homotopică a spaţiilor Hausdorff local compacte) la categoria Ens care asociază fiecărui $X \in \mathrm{Ob}\,(\overset{h}{\mathrm{HLC}})$ mulţimea $L'(X)$ a claselor de cofibrări coechivalente homotopic avînd spaţiul total X şi baza din $\mathrm{Ob}\,(\overset{h}{\mathrm{HLC}})$. Se stabileşte un morfism functorial între L' şi $L \cdot \mathrm{Hom}\,(\cdot\,, Z)$, unde L este functorul contravariant definit în [2].

2. Vom utiliza :

T e o r e m a A [1]. *Fie $f : X' \to X$ o cofibrare cu X' şi X spaţii Hausdorff local compacte. Atunci aplicaţia $p' : Y^X \to Y^{X'}$ (Y un spaţiu topologic oarecare), definită prin $p'(g) = gf$, pentru $g : X \to Y$, este o fibrare Hurewicz (H-fibrare).*

Notăm prin Hom (X, Y) spaţiul topologic Y^X topologizat cu topologia compact deschisă ; în consecinţă vom nota aplicaţia p' de mai sus prin Hom (f, Y). Va fi utilă

P r o p o z i ţ i a 1. *Fie $f : X' \to X$ o cofibrare, X', X Hausdorff local compacte (HLC) şi $F_1, F_2 : X \times I \to Y$ două aplicaţii încît $F_1/X \times 0 = F_2/X \times 0$ şi $F_1(f \times 1) = F_2(f \times 1)$. Atunci $F_1 \underset{f}{\simeq} F_2$ rel $X \times 0$ (adică există o aplicaţie $H : X \times I \times I \to Y$ încît $H(x, t, 0) = F_1(x, t)$, $H(x, t, 1) = F_2(x, t)$, $H(x, 0, t') = F_1(x. 0)$, $H(f(x'), t, t') = F_1(f(x'), t)$.*

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[1]) Prezentată la sesiunea ştiinţifică a Universităţii „Al. I. Cuza" din Iaşi, din 25—27 octombrie 1971.

D e m o n s t r a ţ i e. Fie Hom $(f, Y) : Y^X \to Y^{X'}$. Definim următoarele două drumuri $\omega_1, \omega_2 : I \to Y^X$ şi anume $\omega_1(t)(x) = F_1(x, t)$ şi $\omega_2(t)(x) = F_2(x, t)$. Avem atunci $\omega_1(0)(x) = F_1(x, 0) = F_2(x, 0) = \omega_2(0)(x)$ şi $p\omega_1(t)\, x') = \omega_1(t)(f(x')) = F_1(f(x'), t) = F_2(f(x'), t) = \omega_2(t)(f(x')) = p\omega_2(t)(x')$. Rezultă, [1], că $\omega_1 \underset{p}{\simeq} \omega_2$ rel $\{0\}$ deci există o aplicaţie $h : I \times I \to Y^X$ încît $h(t, 0) = \omega_1(t)$, $h(t, 1) = \omega_2(t)$, $h(0, t') = \omega_1(0) = \omega_2(0)$ şi $h(t, t')(f(x')) = \omega_1(t)(f(x')) = \omega_2(t)(f(x'))$. Putem defini aplicaţia $H : X \times I \times I \to Y$ prin $H(x, t, t') = h(t, t')(x)$ care se verifică imediat că satisface condiţiile propoziţiei.

P r o p o z i ţ i a 2. *Dacă $f : X' \to X$ este o cofibrare cu X' şi X, HLC şi dacă oricare ar fi două aplicaţii $F_1, F_2 : X \times I \to Y$ cu $F_1/X \times 0 = F_2/X \times 0$ şi $F_1(f \times 1) = F_2(f \times 1)$ să rezulte $F_1 = F_2$ atunci fibrarea Hom (f, Y) are proprietatea de drum unic ridicat.*

Demonstraţia este imediată.

Dacă $f_1 : X \to X_1$ este o cofibrare şi $f_2 : X \to X_2$ este o aplicaţie continuă atunci fie X' spaţiul obţinut din suma $X_1 \vee X_2$ identificînd $f_1(x)$ cu $f_2(x)$. Se ştie că aplicaţia $i_2 : X_2 \to X'$ este o cofibrare numită indusă de f_2 în raport cu cofibrarea f_1. Pentru un spaţiu topologic Y, în ipoteza că X, X_1, X_2 sînt HLC, cofibrarea i_2 defineşte o fibrare Hom (i_2, Y). Pe de altă parte putem considera fibrarea Hom (f_1, Y). Dacă considerăm şi aplicaţia Hom (f_2, Y) atunci putem considera fibrarea indusă de Hom (f_2, Y) în raport cu Hom (f_1, Y). Are loc

L e m a 1. *Fibrările Hom (i_2, Y) şi Hom $(f_2, Y)^*$ Hom (f_1, Y) sînt echivalente topologic.*

D e m o n s t r a ţ i e . Fie spaţiul total al fibrării induse de Hom (f_2, Y) $(Y^X)_{\mathrm{Hom}(f_2, Y)} = \{(\alpha, \beta) \in Y^{X_1} \times Y^{X_2} / \mathrm{Hom}\,(f_1, Y)(\alpha) = \mathrm{Hom}\,(f_2, Y)(\beta)$ adică $\alpha f_1(x) = \beta f_2(x)\}$ şi fibrarea indusă definită prin (Hom $(f_2, Y)^*$ Hom (f_1, Y)) $(\alpha, \beta) = \beta$.

Fie $u : (Y^{X_1})_{\mathrm{Hom}(f_2, Y)} \to Y^{X'}$ şi $v : Y^{X'} \to (Y^{X_1})_{\mathrm{Hom}(f_2, Y)}$ difinite astfel: $u(\alpha, \beta)([x_1]) = \alpha(x_1)$, $u(\alpha, \beta)([x_2]) = \beta(x_2)$ şi $v(g) = (gi_1, gi_2)$ unde $i_1 : X_1 \to X'$. Se verifică că aceste aplicaţii sînt bine definite şi că sînt inverse una alteia ceea ce dovedeşte afirmaţia.

D e f i n i ţ i a 1. Două cofibrări $f : X' \to X_1$ şi $f_2 : X' \to X_2$ le vom numi coechivalente homotopic dacă există două aplicaţii $u : X_1 \to X_2$ şi $v : X_2 \to X_1$ încît $uf_1 = f_2$, $vf_2 = f_1$ şi $uv \underset{f_2}{\simeq} 1_{X_2}$ şi $vu \underset{f_1}{\simeq} 1_{X_1}$. Se verifică imediat că aceasta este o relaţie de echivalenţă algebrică.

L e m a 2. *Dacă $f_1 : X' \to X_1$ şi $f_2 : X' \to X_2$ sînt două cofibrări coechivalente homotopic şi X', X_1, X_2 sînt HLC iar Y este un spaţiu topologic oarecare, atunci fibrările Hom (f_1, Y) şi Hom (f_2, Y) sînt echivalente homotopic.*

D e m o n s t r a ţ i e. Fie $u : X_1 \to X_2$ şi $v : X_2 \to X_1$ realizînd coechivalenţa homotopică a cofibrărilor date. Atunci Hom (u, Y) şi Hom (v, Y) realizează echivalenţa homotopică a fibrărilor Hom (f_1, Y) şi Hom (f_2, Y), cum se poate verifica direct.

P r o p o z i ţ i a 3. *Fie $f : X \to Y$ o cofibrare cu X şi Y HLC şi $g' \simeq g'' : X \to Z$. Atunci cofibrările induse de g' şi g'' în raport cu f sînt coechivalente homotopic.*

D e m o n s t r a ţ i e. Fie $f' : Z \to X'$ şi $f'' : Z \to X''$ cofibrările induse de g' şi respectiv g'' iar $i' : Y \to X'$ şi $i'' : Y \to X''$ aplicaţiile canonice. Vom nota prin [ ], elementele din X' şi prin { } elementele din X''. Observăm că avem $i'f = f'g'$ şi analog $i''f = f''g''$. Să considerăm diagrama

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f\circ F"'] & Y \arrow[d, "X'"] \arrow[dl, "i'"'] \\ X\times I \arrow[r, "f\times 1"] & Y\times I \end{tikzcd}$$

unde $F : X \times I \to Z$ cu $F(x, 0) = g'(x)$ şi $F(x, 1) = g''(x)$. Avem $f'(F(x, 0)) = f'g'(x) = i'f(x)$ deci există $F' : Y \times I \to X'$ cu $F'(y, 0) = i'(y) = [y]$ şi $F'(f(x), t) = f'(F(x, t))$. În mod analog există $F'' : Y \times I \to X''$ cu $F''(y, 1) = \{y\}$ şi $F''(f(x), t) = f''(F(x, t))$. Să considerăm acum aplicaţia $u : X' \to X''$ definită prin : $u([y]) = F''(y, 0)$ şi $u([z]) = \{z\}$ care este bine definită şi analog definim $v : X'' \to X'$ prin $v(\{y\}) = F'(y, 1)$ şi $v(\{z\}) = [z]$. Se verifică că $uf' = f''$ şi $vf'' = f'$. Să considerăm acum $uF'(f \times 1)(x, t) = uF'(f(x), t) = uf'(F(x, t)) = f''(F(x, t)) = F''(f(x), t) = F''(f \times 1)(x, t)$ deci $uF'(f \times 1) = F''(f \times 1)$. În plus $uF'(y, 0) = u[y] = F''(y, 0)$ deci $uF'/Y \times 0 = F''/Y \times 0$. Din propoziţia 1 rezultă că

$$uF' \underset{f}{\simeq} F'' \operatorname{rel} Y \times 0. \tag{1}$$

În mod analog avem

$$vF'' \underset{f}{\simeq} F' \operatorname{rel} Y \times 1. \tag{2}$$

Din relaţiile (1) şi (2) rezultă

$$uvF'' \underset{f}{\simeq} F'' \operatorname{rel} Y \times 0 \tag{3}$$

şi

$$vuF' \underset{f}{\simeq} F' \operatorname{rel} Y \times 1. \tag{4}$$

Relaţia (3) înseamnă că există o homotopie $H : Y \times I \times I \to X''$ cu $H(y, t, 0) = uvF''(y, t)$; $H(y, t, 1) = F''(y, t)$; $H(y, 0, t) = uvF''(y, 0) = F''(y, 0)$ şi $H(f(x), t, t') = F''f((x), t) = uvF''(f(x), t)$. Definim acum aplicaţia $h : X'' \times I \to X''$ cu $h(\{y\}, t) = H(y, 1, t)$ şi $h(\{z\}, t) = \{z\}$ care se verifică că este bine definită şi avem $h : uv \underset{f''}{\simeq} 1$.

În mod analog, utilizînd relaţia (4) se arată că $vu \underset{f'}{\simeq} 1$ şi deci cofibrările f' şi f'' sînt coechivalente homotopic.

Să notăm acum prin $L'(X)$ mulţimea claselor de cofibrări coechivalente homotopic avînd spaţiul total X şi spaţiul bază un spaţiu HLC. Putem demonstra acum

Teorema 1. *Există un factor covariant* $L' : (\overset{h}{HLC}) \to Ens$ *care asociază unui spaţiu Hausdorff local compact* X *mulţimea* $L'(X)$.

Demonstraţie. Fie $f : X \to Y$ o cofibrare încît $[f] \in L'(X)$ şi fie $g : X \to Z$ o aplicaţie continuă între spaţii HLC. Definim atunci aplicaţia de mulţimi $L'(g) : L'(X) \to L'(Z)$ şi anume $f \to g_* f$(cofibrarea indusă de aplicaţia g în raport cu cofibrarea f). Fie acum $f' : X \to Y'$ o cofibrare coechivalentă cu f. Să arătăm că $g_* f$ şi $g_* f'$ sînt coechivalente homotopic Avem deci diagramele

$$\begin{tikzcd} & Y \arrow[d, shift right, "u"'] \\ X \arrow[ur, "f"] \arrow[r, "f'"] \arrow[dr, "g"'] & Y' \arrow[u, shift right, "v"'] \\ & Z \end{tikzcd}$$

cu $uf = f'$, $vf' = f$, $uv \underset{f'}{\simeq} 1$ şi $vu \underset{f}{\simeq} 1$. Fie cofibrările induse de aplicaţia g pentru f şi respectiv $f' : i : Z \to X'$ şi $i' : Z \to X''$ unde spaţiul X' se obţine din suma $Y \vee Z$ prin identificarea lui $f(x)$ cu $g(x)$ iar spaţiul X'' se obţine din suma $X' \vee Z$ prin identificarea lui $f'(x)$ cu $g(x)$. Definim acum aplicaţiile $U : X' \to X''$ şi $V : X'' \to X'$ prin $U[y] = \{u(y)\}$ şi $U[z] = \{z\}$ şi $V([y']) = \{v(y')\}$ şi $V([z]) = \{z\}$ care sînt bine definite.

Avem în plus $U(g_* f)(z) = Ui(z) = U([z]) = \{z\}$ şi $i'(z) = \{z\}$ deci $Ui = i'$. În mod analog $Vi'(z) = V\{z\} = [z]$ deci $Vi' = i$. Apoi $VU[y] = [vu(y)]$ şi $VU[z] = [z]$. Să considerăm acum aplicaţia $H_t : X' \to X'$ definită bine prin $H_t([y]) = [h_t(y)]$ şi $H_t[z] = [z]$. Avem $H_0 = VU$ şi $H_1 = 1$. În plus $H_t i = i$ adică avem $H_t : VU \underset{i}{\simeq} 1$. În mod analog se arată că $UV \underset{i'}{\simeq} 1$ şi deci i şi i' sînt două cofibrări coechivalente homotopic şi deci putem defini aplicaţia $L'(g) : L'(X) \to L'(Y)$ prin $L'(g)[f] = [g_* f]$.

Dacă acum $g = 1 : X \to X$ atunci avem imediat că, cofibrarea $1_* f$ este coechivalentă homotopic cu f şi deci $L'(1)[f] = [f]$ adică $L'(1) = 1_{L'(.)}$. Fie acum $f : X \to Y$ cu $[f] \in L'(X)$ şi fie $g : X \to Z$ şi $g' : Z \to Q$ două aplicaţii continue între spaţii HLC. Fie atunci $g_* f : Z \to X'$ cofibrarea indusă de g în raport cu cofibrarea f : $g'_*(g_* f) : Q \to Z'$ cofibrarea indusă de g' în raport cu cofibrarea $g_* f$ şi $(g' g)_* f : Q \to X''$ cofibrarea indusă de aplicaţia $g'g$ în raport cu cofibrarea f. Vom nota prin [ ], { }, şi < > elementele în spaţiile X', Z' şi respectiv X''. Spaţiul X' se obţine din suma $Y \vee Z$ identificînd $f(x)$ cu $g(x)$. Spaţiul Z' se obţine din suma $X' \vee Q$ identificînd $g'(z)$ cu $(g_* f)(z)$ iar spaţiul X'' este obţinut din suma $Y \vee Q$ prin identificarea elementelor $f(x)$ şi $g'g(x)$. Să arătăm că cofibrările $g'_*(g_* f)$ şi $(g' g)_* f$ sînt coechivalente homotopic. Va rezulta atunci că $L'(g') L'(g) = L'(g' g)$.

Fie pentru aceasta aplicaţia $\alpha : Z' \to X''$ definită în felul următor:

$\alpha\{[y]\} = < y >$ pentru $y \in Y$

$\alpha\{[z]\} = < g'(z) >$ pentru $z \in Z$ şi

$\alpha\{q\} = < q >$ pentru $q \in Q$.

Se arată că această aplicaţie este bine definită considerînd următoarele cazuri:

a) $\{[y]\} = \{[y_1]\}$; b) $\{[z]\} = \{[z_1]\}$; c) $\{q\} = \{q_1\}$;
d) $\{[y]\} = \{[z]\}$; e) $\{[y]\} = \{q\}$ şi f) $\{[z]\} = \{q\}$.

Definim şi aplicaţia $\beta : X'' \to X'$ în felul următor:

$\beta(< y >) = \{[y]\}$ pentru $y \in Y$ şi
$\beta(< q >) = \{q\}$ pentru $q \in Q$.

Se arată că această aplicaţie este bine definită considerînd cazurile:
a) $< y > = < y_1 >$; b) $< q > = < q_1 >$ şi c) $< y > = < q >$.
Avem acum $\beta(g'g)_* f(q) = \beta(< q >) = \{q\} = g'_*(g_* f)(q)$. În sfîrşit $\alpha\beta(< y >) = \alpha\{[y]\} = < y >$; $\alpha\beta(< q >) = \alpha\{q\} = < q >$ şi deci $\alpha\beta = 1$.

Analog $\beta\alpha\{[y]\} = \beta(< y >) = \{[y]\}$; $\beta\alpha\{[z]\} = \beta(< g'(z) >) = \{g'(z)\} = \{[z]\}$ şi $\beta\alpha\{q\} = \beta(< q >)$ de unde rezultă că $\beta\alpha = 1$. Dacă ţinem cont de Propoziţia 3 rezultă că pentru un morfism g în categoria $(\overset{h}{\mathrm{HLC}})$ putem defini $L'([g])[f] = [g_* f]$ şi cum am văzut este definit astfel un functor covariant, ceea ce demonstrează teorema.

Să considerăm acum functorul covariant $L : \overset{h}{\mathrm{Top}} \to \mathrm{Ens}$ care asociază unui spaţiu topologic X mulţimea $L(X)$ a claselor de H-fibrări echivalente homotopic şi care au spaţiul bază X, iar unui morfism $f : X \to Y$ i se asociază aplicaţia $L[f] : L(Y) \to L(X)$ definită prin $L[f][p] = [f^* p]$ (fibrarea indusă de f pentru p), [2].

Pentru un spaţiu topologic Z definim un functor covariant $LZ : (\overset{h}{\mathrm{HLC}}) \to \mathrm{Ens}$ şi anume $LZ(X) = L(Z^X)$ şi pentru $f : X \to Y$, un morfism în $(\overset{h}{\mathrm{HLC}})$, definim $f_Z = \mathrm{Hom}(f, Z)$. În adevăr dacă $f = 1$ atunci definind $LZ(f) = L(f_Z)$ avem $LZ(1) = L(1_Z) = 1_{LZ(X)}$. Dacă $f : X_1 \to X_2$ şi $g : X_2 \to X_3$ atunci $LZ(gf) = LZ(g)\, LZ(f)$. În plus dacă $f \simeq g : X \to Y$ atunci $f_Z \simeq g_Z$ cum se verifică imediat. Deci avem functorul covariant $LZ = L.\mathrm{Hom}(., Z)$ $LZ[f][p] = [\mathrm{Hom}(f, Z)^* p]$.

P r o p o z i ţ i a 4. *Pentru fiecare spaţiu topologic Z există un morfism functorial între functorii L', $L \circ Hom(.\ , Z) : (\overset{h}{HLC}) \to Ens$.*

D e m o n s t r a ţ i e Vom defini un morfism functorial $Z^{\cdot} : L' \to L \circ \mathrm{Hom}(.\ , Z)$ şi anume pentru un spaţiu HLC definim aplicaţia $Z^{\cdot}(X)$:

: $L'(X) \to L(Z^X)$ prin $Z^{\cdot}(X)[f] = [\mathrm{Hom}\,(f, Z)]$ (lema 2) şi avem, pentru $g : X \to Y$ următoarea diagramă comutativă :

$$\begin{tikzcd} L'(X) \arrow[r, "Z^{\cdot}(X)"] \arrow[d, "L'([g])"'] & LZ(X) \arrow[d, "LZ([g])"] \\ L'(Y) \arrow[r, "Z^{\cdot}(Y)"] & LZ(Y) \end{tikzcd}$$

În adevăr avem

$$LZ([g])\, Z^{\cdot}(X)[f] = LZ([g])\,[f_Z] = [g^* f_Z]$$

şi

$$Z^{\cdot}(Y) L'([g])\,([f]) = Z^{\cdot}(Y)\,([g_* f]) = [(g_* f)_Z].$$

După lema 1 rezultă că fibrările $g^* f_Z$ şi $(g^* f)_Z$ sînt echivalente homotopic şi deci $[g^* f_Z] = [(g_* f)_Z]$ ceea ce dovedeşte afirmaţia propoziţiei.

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COFIBRATION MAPS BETWEEN HAUSDORFF LOCALLY COMPACT SPACES

(Summary)

In this paper homotopic coequivalence of cofibrations is defined (definition 1). Theorem 1 proves that there exists a covariant functor L' which associates to a locally compact Hausdorff space X the set $L'(X)$ of homotopical coequivalence cofibrations classes having the total space X and a locally compact Hausdorff base space.

A connection with Hurewicz fiber spaces is also given (proposition 4).

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ASUPRA UNEI CLASE DE ALGEBRE CHIMICE DE ORDINUL PATRU

DE

ION ENESCU şi IOLANDA ENESCU

1. În [1] se arată că oricărui sistem de ecuaţii diferenţiale ordinare cu coeficienţi reali de forma

$$x^i = a^i_{jk}\, x^j x^k\,, \qquad (i,\ j,\ k = 1,\ 2\,,\ldots,\ n), \tag{1}$$

denumit sistem diferenţial pătratic, i se poate asocia, în mod unic, o algebră reală de ordin n, pentru care, într-o bază $e_1,\ e_2\,,\ldots,\ e_n$, produsul este definit prin

$$e_j\, e_k = a^i_{jk}\, e_i\,, \qquad (i,\ j,\ k = 1,\ 2\,,\ldots,\ n), \tag{2}$$

constantele de structură a^i_{jk} fiind tocmai coeficienţii ce intervin în (1) şi care, datorită formei pătratice în care apar, sînt supuşi condiţiei

$$a^i_{jk} = a^i_{kj}\,. \tag{3}$$

Se vede deci că algebra (2) asociată sistemului (1) este comutativă, dar, în general, neasociativă.

Pe de altă parte, în [2] se arată că cinetica reacţiilor chimice de ordin doi se poate exprima printr-un sistem de forma (1) şi, în consecinţă i se poate asocia o algebră (2).

2. În această Notă ne propunem studiul algebrelor asociate, în sensul de mai sus, la două reacţii chimice consecutive (în serie-paralele) a căror cinetică este dată de sistemul [3] :

$$
(4) \qquad \begin{cases} \dot{x}_1 = -k_1 x_1 x_2, \\ \dot{x}_2 = -k_1 x_1 x_2 - k_2 x_2 x_3, \\ \dot{x}_3 = k_1 x_1 x_2 - k_2 x_2 x_3, \\ \dot{x}_4 = k_2 x_2 x_3, \end{cases} \qquad (k_1,\ k_2 > 0),
$$

unde k_1, k_2 sînt constantele vitezei de reacţie.

Exemple de asemenea reacţii pot fi găsite în halogenarea (sau nitrarea) prin substituţia excesivă a hidrocarburilor precum şi în adiţia oxizilor alchenici la compuşi din clasa donorilor de protoni [3].

Algebra A asociată sistemului (4) are, într-o bază $e_1, \ldots, e_4$, tabela:

$$
(5) \qquad \begin{array}{c|cccc}
 & e_1 & e_2 & e_3 & e_4 \\
\hline
e_1 & 0 & -\frac{k_1}{2}(e_1 + e_2 - e_3) & 0 & 0 \\
e_2 & -\frac{k_1}{2}(e_1 + e_2 - e_3) & 0 & -\frac{k_2}{2}(e_2 + e_3 - e_4) & 0 \\
e_3 & 0 & -\frac{k_2}{2}(e_2 + e_3 - e_4) & 0 & 0 \\
e_4 & 0 & 0 & 0 & 0
\end{array},
$$

unde parametrii reali k_1 şi k_2 sînt strict pozitivi.

Cu schimbarea de bază

$$
(6) \qquad u_i = -\frac{2}{k_2} e_i, \qquad (i = 1, \ldots, 4),
$$

prin calcule evidente, făcute conform tabelei (5), se obţine în baza (6), pentru A, tabela de înmulţire

$$
(7) \qquad \begin{array}{c|cccc}
 & u_1 & u_2 & u_3 & u_4 \\
\hline
u_1 & 0 & \alpha(u_1 + u_2 - u_3) & 0 & 0 \\
u_2 & \alpha(u_1 + u_2 - u_3) & 0 & u_2 + u_3 - u_4 & 0 \\
u_3 & 0 & u_2 + u_3 - u_4 & 0 & 0 \\
u_4 & 0 & 0 & 0 & 0
\end{array},
$$

în care

$$
(8) \qquad \alpha = \frac{k_1}{k_2},
$$

ceea ce demonstrează

T e o r e m a 1. *Algebra A depinde numai de raportul* $\frac{k_1}{k_2}$.

Schimbînd baza e_i prin

$$u'_i = -\frac{2}{k_1} e_i, \qquad (i = 1, \ldots, 4), \tag{9}$$

se obţine pentru A tabela

(10)

	u'_1	u'_2	u'_3	u'_4
u'_1	0	$u'_1 + u'_2 - u'_3$	0	0
u'_2	$u'_1 + u'_2 - u'_3$	0	$\frac{1}{\alpha}(u'_2 + u'_3 - u'_4)$	0
u'_3	0	$\frac{1}{\alpha}(u'_2 + u'_3 - u'_4)$	0	0
u'_4	0	0	0	0

Cum algebra (5) este izomorfă cu respectiv (7) şi (10), urmează, ţinînd seama de tranzitivitatea izomorfismului, că algebrele date prin respectiv (7) şi (10) sînt izomorfe.

Transformarea bazei din (7), definită prin

$$w_i = \frac{k_2}{k_1} u_i, \qquad (i = 1, \ldots, 4), \tag{11}$$

conduce la tabela

	w_1	w_2	w_3	w_4
w_1	0	$w_1 + w_2 - w_3$	0	0
w_2	$w_1 + w_2 - w_3$	0	$\frac{1}{\alpha}(w_2 + w_3 - w_4)$	0
w_3	0	$\frac{1}{\alpha}(w_2 + w_3 - w_4)$	0	0
w_4	0	0	0	0

ceea ce dovedeşte

T e o r e m a 2. *Toate algebrele A corespunzătoare valorilor α ale parametrului sînt respectiv izomorfe cu acelea corespunzătoare valorilor* $\frac{1}{\alpha}$. *Schimbarea de bază care dă trecerea de la cazul α la respectiv* $\frac{1}{\alpha}$ *este dată de* (11).

Teorema 2 reduce studiul algebrelor A numai la acelea corespunzătoare lui $\alpha \in (0, 1]$, avînd tabela de înmulţire (7).

Fie deci $\alpha \in (0, 1]$. Schimbarea de bază

$$v_1 = u_1 + u_2 - u_3, \quad v_2 = u_2 + u_3 - u_4, \quad v_3 = u_3, \; v_4 = u_4, \tag{12}$$

conduce la tabela

(13)

	v_1	v_2	v_3	v_4
v_1	$2(\alpha v_1 - v_2)$	αv_1	v_2	0
v_2	αv_1	$2v_2$	v_2	0
v_3	v_2	v_2	0	0
v_4	0	0	0	0

care arată că, pentru determinarea tipurilor neizomorfe corespunzătoare valorilor $\alpha \in (0, 1]$ ale parametrului, este necesar şi suficient să se studieze numai idealul $\mathcal{J}$, determinat de v_1, v_2, deci algebra de ordin doi dată prin

(14)

	v_1	v_2
v_1	$2(\alpha v_1 - v_2)$	αv_1,
v_2	αv_1	$2v_2$

, $\alpha \in (0, 1]$.

3. Să determinăm pentru $\mathcal{J}$, elementele idempotente. Ecuaţia $E^2 = E$ arată că singurii idempotenţi proprii ai lui $\mathcal{J}$ sînt

$$E_1 = \frac{1}{2} v_2 \quad \text{şi} \quad E_2 = \frac{1-\alpha}{2\alpha(2-\alpha)} v_1 + \frac{1}{2\alpha(2-\alpha)} v_2 . \tag{15}$$

Pentru $\alpha = 1$, se vede imediat că $E_1 = E_2$, acesta fiind singurul caz pentru care cei doi idempotenţi coincid, fapt ce stabileşte

T e o r e m a 3. *Idealul $\mathcal{J}$ corespunzător valorii $\alpha = 1$ a parametrului este neizomorf cu oricare altul obţinut pentru $\alpha \in (0, 1)$.*

Notînd acest tip cu $\mathcal{J}_1$, tabela sa este

(16)

	v_1	v_2
v_1	$2v_1 - 2v_2$	v_1 .
v_2	v_1	$2v_2$

Avem imediat şi

T e o r e m a 4. *Algebra A, corespunzătoare lui $\alpha = 1$, este neizomorfă cu oricare alta, obţinută pentru valori ale parametrului $\alpha \neq 1$.*

În continuare considerăm $\alpha \in (0, 1)$.

Cercetarea asociativităţii produsului în $\mathcal{J}$ arată că această proprietate nu poate avea loc pentru nici o valoare a lui α conţinută în $(0, 1)$ şi în consecinţă rezultă

T e o r e m a 5. *Algebrele A nu sînt asociative pentru nici o valoare a parametrului α.*

Se pune, în continuare, problema determinării tipurilor neizomorfe de algebre A, care, evident, se reduce la determinarea tipurilor neizomorfe ale idealului $\mathcal{J}$. În acest scop să facem în $\mathcal{J}$ schimbarea de bază

(17) $$\varepsilon_1 = v_1 - v_2, \quad \varepsilon_2 = v_1 + v_2,$$

ceea ce conduce la tabela

(18)

	ε_1	ε_2
ε_1	0	$(\alpha+2)\varepsilon_1 + (\alpha-2)\varepsilon_2$
ε_2	$(\alpha+2)\varepsilon_1 + (\alpha-2)\varepsilon_2$	$2\alpha(\varepsilon_1+\varepsilon_2)$

care, aşa cum se vede direct, pune în evidenţă elementul nilpotent ε_1.

În baza (18), elementele idempotente (15) au respectiv forma

(15′) $$E_1 = \frac{1}{4}(\varepsilon_2 - \varepsilon_1) \quad \text{şi} \quad E_2 = \frac{1}{4(\alpha-2)}\varepsilon_1 + \frac{1}{4\alpha}\varepsilon_2.$$

Fie acum α_1 şi α_2 două valori ale parametrului α, fixate arbitrar în (0, 1) şi să notăm prin $E_1^{(1)}$, E_2^{1}, $E_1^{(2)}$, $E_2^{(2)}$ idempotenţii (15′) corespunzători respectiv lui α_1 şi α_2, deci

$$E_1^{(1)} = \frac{1}{4}(\varepsilon_2 - \varepsilon_1), \quad E_2^{(1)} = \frac{1}{4(\alpha_1-2)}\varepsilon_1 + \frac{1}{4\alpha_1}\varepsilon_2,$$

$$E_1^{(2)} = \frac{1}{4}(\varepsilon_2 - \varepsilon_1), \quad E^{(2)} = \frac{1}{4(\alpha_2-2)}\varepsilon_1 + \frac{1}{4\alpha_2}\varepsilon_2.$$

Căutăm o funcţie φ care să reprezinte izomorf idealul $\mathcal{J}^{(1)}$ corespunzător valorii α_1, pe idealul $\mathcal{J}^{(2)}$ corespunzător valorii α_2.

Să notăm $\varphi(u_1) = a\varepsilon_1 + b\varepsilon_2$. Ţinînd seama că prin izomorfism elementele nilpotente se corespund, urmează că

$$\varphi^2(u_1) = \varphi(u_1)\,\varphi(u_1) = 0, \quad \text{adică} \quad (a\varepsilon_1 + b\varepsilon_2)^2 = 0.$$

Această condiţie conduce, după efectuarea calculelor, la

(19) $$\varphi(u_1) = a\varepsilon_1,$$

unde a este un scalar real arbitrar.

Folosind acum proprietatea izomorfismului de a conserva şi elementele idempotente, urmează că funcţia φ trebuie să satisfacă una din următoarele două grupe de condiţii:

(20) $$\varphi(E_1^{(1)}) = E_1^{(2)}, \quad \varphi(E_2^{(1)}) = E_2^{(2)},$$

(21) $$\varphi(E_1^{(1)}) = E_2^{(2)}, \quad \varphi(E_1^{(2)}) = E_1^{(2)}.$$

Transcrise în baza (17), aceste condiţii devin, respectiv:

(20′) $$\begin{cases} -\frac{1}{4}\varphi(\varepsilon_1) + \frac{1}{4}\varphi(\varepsilon_2) = -\frac{1}{4}\varepsilon_1 + \frac{1}{4}\varepsilon_2, \\ \frac{1}{4(\alpha_1-2)}\varphi(\varepsilon_1) + \frac{1}{4\alpha_1}\varphi(\varepsilon_2) = \frac{1}{4(\alpha_2-2)}\varepsilon_1 + \frac{1}{4\alpha_2}\varepsilon_2, \end{cases}$$

$$(21') \quad \begin{cases} -\frac{1}{4}\varphi(\varepsilon_1) + \frac{1}{4}\varphi(\varepsilon_2) = \frac{1}{4(\alpha_2-2)}\varepsilon_1 + \frac{1}{4\alpha_2}\varepsilon_2, \\ \frac{1}{4(\alpha_1-2)}\varphi(\varepsilon_1) + \frac{1}{4\alpha_1}\varphi(\varepsilon_2) = -\frac{1}{4}\varepsilon_1 + \frac{1}{4}\varepsilon_2. \end{cases}$$

Utilizînd (19), condiţiile (20′) conduc la $\alpha_1 = \alpha_2$, adică la izomorfismul identic, iar condiţiile (21′) impun relaţia $\alpha_1\alpha_2 = 1$, ceea ce, evident, nu este posibil, întrucît α_1 şi α_2 sînt subunitari.

Aceste rezultate stabilesc

T e o r e m a 6. *Toate idealele $\mathcal{J}$ corespunzătoare valorilor $\alpha \in (0, 1)$ ale parametrului, sînt neizomorfe.*

Teorema 6 rezolvă problema determinării tipurilor neizomorfe de algebre (5), prin

T e o r e m a 7. *Există o infinitate de tipuri neizomorfe de algebre A, corespunzătoare valorilor $\alpha \in (0, 1)$. Tipul $\alpha = 1$ este neizomorf cu oricare altul şi el este singurul caz reprezentat printr-o singură algebră A. Toate celelalte se grupează în perechi după, respectiv, α şi $\frac{1}{\alpha}$ fiecare pereche (A_α, $A_{1/\alpha}$) reprezentînd cîte un tip neizomorf de algebră A.*

4. Fără a reda calculele dificile ce au fost făcute, semnalăm faptul că nici una din algebrele (5) nu este respectiv alternativă, Jordan sau cu puteri asociative.

5. În încheiere menţionăm faptul că reacţiile chimice a căror cinetică are forma (4) se pot clasifica după tipurile neizomorfe de algebre (5) asociate lor, aşa cum se propune în [2].

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SUR UNE CLASSE D'ALGÈBRES CHIMIQUES D'ORDRE QUATRE

(Résumé)

En considérant deux réactions chimiques consécutives dont la cinétique est donnée par le système (4), on détermine les types non-isomorphes d'algèbres associées (5), en mettant aussi en évidence leurs propriétés de structure.

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CORESPONDENŢE ÎNTRE FAMILII DE VARIETĂŢI R, ÎN SENS MAYER

DE

ŞTEFANIA RUSCIOR

1. În această notă se consideră corespondenţe prin varietăţi tangente parţial paralele, între două familii de varietăţi R, din S_n.

O. Mayer a studiat într-un număr de 6 lucrări suprafeţe riglate cu titlul *Sur les surfaces réglées*, publicate între 1928—1944 în [1]...[6].

O. Mayer numeşte *o familie* R, ∞^1 curbe aşezate pe o suprafaţă riglată astfel încît prin fiecare punct al unei generatoare g de pe riglată trece o singură curbă aparţinînd familiei R.

2. Se extinde aici această problemă în S_n afin, considerînd familii de varietăţi R în sens Mayer, ∞^1 varietăţi de ordin k aşezate pe o varietate riglată (V) de ordin $k+1$, astfel încît prin fiecare punct al unei generatoare de pe (V) să treacă o varietate R din familia considerată. Fie varietatea riglată V de tipul (V_0^k) adică o varietate riglată care nu conţine nici un punct singular pe generatoarea ei generică şi care are secţiunea improprie de dimensiune K [7].

Ecuaţia unei astfel de varietăţi (V) este:

$$\bar{r} = \bar{\rho}(u_i) + w\bar{\alpha}(u_i), \qquad (i = 1, \ldots, k). \tag{1}$$

Varietatea $\bar{\rho} = \bar{\rho}(u_i)$ reprezintă o varietate directoare cu k dimensiuni, iar $\bar{\alpha} = \bar{\alpha}(u_i)$ versorul generatoarei determinat în fiecare punct al varietăţii directoare.

Secţiunea improprie este de dimensiune k dacă şi numai dacă matricea funcţională $\| \bar{\alpha}_{u_1}, \bar{\alpha}_{u_2}, \ldots \bar{\alpha}_{u_k}, \bar{\alpha} \|$ are rangul $k+1$.

Presupunem în cele ce urmează condiţii de regularitate pentru funcţiile ce definesc varietatea directoare şi versorul generatoarei.

În aceste condiţii familia R de pe (V) este caracterizată prin $w =$ $=$ const. Ea este formată din varietăţi de dimensiune k cu următoarele proprietăţi :

a) Prin fiecare punct al generatoarei (g) trece o varietate R determinată de o valoare dată lui w care reprezintă coordonata proiectivă a unui punct de pe (g). Există deci o corespondenţă biunivocă între familia R şi punctele generatoarei.

b) Varietăţile R au proprietatea de a tăia două generatoare g_1 şi g_2 de pe (V) în puncte ce se corespund omografic (prin egalitatea coordonatei proiective w).

c) Dacă se consideră patru varietăţi R din familia precedentă ele intersectează generatoarele riglatei V în patru puncte care au acelaşi biraport.

d) De aici rezultă că dacă se cunosc trei varietăţi R (aparţinînd familiei R) întreaga familie R este determinată, deoarece orice altă varietate R este locul punctelor care împreună cu cele trei puncte determinate de cele 3 varietăţi prin intersecţie cu o generatoare generică (g) formează un biraport constant.

3. În cele ce urmează se studiază corespondenţe între două familii de varietăţi R astfel ca în punctele corespunzătoare, varietăţile liniare tangente duse să fie parţial paralele, şi suma dimensiunilor varietăţilor R puse în corespondenţă să fie egală cu dimensiunea spaţiului ambiant. Astfel de corespondenţe nu se pot stabili în spaţii de dimensiune impară, deoarece dimensiunea varietăţilor R este $1/2$ din dimensiunea spaţiului ambiant şi în acelaşi timp un număr întreg ; prin urmare spaţiul în care vom trata problema este de dimensiune $2n$. Într-un astfel de spaţiu există mai multe categorii de paralelisme parţiale caracterizate de funcţia numerică introdusă într-o lucrare anterioară [8], în conformitate cu care dimensiunea paralelismului δ între 2 varietăţi A, B este :

$$\delta = \frac{\mathrm{d}\,[A \cap B] + 1}{\min\{\mathrm{d}\,[A], \mathrm{d}\,[B]\}} .$$

Deoarece în problema considerată intervine un paralelism parţial între 2 varietăţi de aceeaşi dimensiune n, δ poate lua $(n-1)$ valori : $\frac{1}{n}$, $\frac{2}{n}, \ldots, \frac{n-1}{n}$; avem deci $(n-1)$ forme de paralelism parţial.

Vom considera în cele ce urmează cazul paralelismului parţial de dimensiune $\delta = \frac{1}{n}$, cazul cînd varietăţile tangente parţial paralele au un singur punct impropriu comun.

Fie o familie R de pe V dată de ecuaţia $\bar{r} = \bar{\rho}(u_i) + w\,\bar{\alpha}(u_i)$ cu $w =$ $=$ const şi $i = 1, \ldots, n$.

O varietate R are dimensiunea n iar varietatea liniară tangentă T dusă într-un punct de pe (g) este determinată de vectorii $\bar{\rho}_{u_i} + w\,\bar{\alpha}_{u_i}$, prin urmare are dimensiunea n şi este determinată în fiecare punct de pe (g) deoarece am presupus varietatea V de tip (V_0^n).

Considerînd o altă varietate riglată V^*, tot de dimensiune n, familia R^* are ecuaţia $\bar{r}^* = \bar{\rho}^*(u_i) + w^*\,\bar{\alpha}^*(u_i)$, cu $w^* = \text{const.}$ şi $i = 1, \ldots, n$.

Între varietăţile V şi V^* există o corespondenţă a generatoarelor g şi g^* pusă în evidenţă prin alegerea parametrilor u_i.

Varietatea tangentă T^* dusă la o varietate R^* este determinată în fiecare punct al generatoarei g^* de vectorii $\bar{\rho}^*_{u_i} + \bar{w}^*\bar{\alpha}^*_{u_i}$.

Cele două varietăţi tangente T şi T^* au un punct impropriu comun dacă determinantul D de ordinul $2n$ este nul, adică :

$$D = [\bar{\rho}_{u_i} + w\,\bar{\alpha}_{u_i},\ \bar{\rho}^*_{u_i} + w^*\bar{\alpha}_{u_i}] = 0.$$

Dezvoltarea acestui determinant conduce la o relaţie algebrică de forma

$$A_{nn}\,w^n\,w^{*n} + A_{n,n-1}\,w^n\,w^{*n-1} \ldots + A_{1,0}\,w + A_{0,1}\,w^* + A = 0, \tag{2}$$

unde coeficienţii A_{lh} sînt determinanţi de ordinul $2n$, bine determinaţi. Obţinem următorul rezultat :

I. *Corespondenţa prin varietăţi liniare tangente parţial paralele de dimensiune $1/n$ considerată între două familii de varietăţi R de dimensiune n şi cu generatoare corespunzătoare într-un S_{2n} afin, determină între punctele generatoarelor corespunzătoare o corespondenţă algebrică de grad $2n$ în raport cu ambele coordonate proiective şi de grad n în raport cu fiecare din coordonate în parte.*

Astfel unui punct de pe g îi corespund n puncte de pe g^* în care varietăţile tangente duse la R şi R^* sînt parţial paralele de dimensiune $1/n$. Deoarece $\exists$ o corespondenţă biunivocă între punctele unei generatoare şi varietăţile familiei R rezultă că :

II. *La o varietate R din familia R îi corespund n varietăţi R^* din familia R^* prin paralelismul parţial considerat.*

4. În S_4 se poate da şi o interpretare geometrică rezultatului obţinut. În acest caz $n = 2$ şi familia R este o familie de suprafeţe.

Să considerăm o hipersuprafaţă riglată de tip (V_0^2) cu ecuaţia $\bar{r} = \bar{\rho}(u, v) + w\,\bar{\alpha}(u, v)$; familia R de pe această varietate este dată de $w = \text{const.}$

Fie V^* a doua hipersuprafaţă riglată de acelaş tip cu V, cu ecuaţia $\bar{r}^* = \bar{\rho}^*(u, v) + w^*\bar{\alpha}^*(u, v)$; familia R^* este caracterizată prin $w^* = \text{const.}$ Alegerea parametrilor pe varietăţile considerate evidenţiază o corespondenţă între generatoarele de pe V şi V^*.

Ne propunem să găsim o corespondenţă între suprafeţele familiei R şi familiei R^* astfel ca în punctele corespunzătoare, planele tangente

duse respectiv la R şi R^* să fie parţial paralele de dimensiune $\delta = \frac{1}{2}$ (semiparalele). Planele tangente T şi T^* duse la o suprafaţă R respectiv R^* au ecuaţiile:

$$\bar{R} = \bar{r} + \lambda \bar{r}_u + \mu \bar{r}_v \quad \text{şi} \quad \bar{R}^* = \bar{r}^* + \lambda^* \bar{r}^*_u + \mu^* \bar{r}^*_v .$$

Aceste plane sînt semiparalele dacă au un punct comun în hiperplanul impropriu. Această condiţie se exprimă analitic prin anularea determinantului $D = [\bar{r}_u, \bar{r}_v, \bar{r}^*_u, \bar{r}^*_v] = 0$, deci: $[\bar{\rho}_u + w\,\bar{\alpha}_u, \bar{\rho}_v + w\,\bar{\alpha}_v, \bar{\rho}^*_u + w^*\,\bar{\alpha}^*_u, \bar{\rho}^*_v + w^*\,\bar{\alpha}^*_v] = 0$.

Ordonînd după w şi w^* obţinem următoarea relaţie algebrică:

$$A_{22}\, w^2 w^{x^2} + A_{21}\, w^2 w^x + A_{12}\, w + \ldots\ldots + A_{1,0}\, w + A_{0,1}\, w^* + A = 0, \tag{3}$$

unde $A_{22} = [\bar{\alpha}_u, \bar{\alpha}_v, \bar{\alpha}^*_u, \bar{\alpha}^*_v] \ldots\ldots A = [\bar{\rho}_u, \bar{\rho}_v, \bar{\rho}^*_u, \bar{\rho}^*_v]$. Avem:

III. *Corespondenţa prin plane tangente semiparalele între familiile R şi R^* de pe două varietăţi riglate V şi V^* de tip (V_0^2) cu generatoare g şi g^* corespunzătoare, determină între punctele generatoarelor g şi g^* o corespondenţă algebrică de grad 4 în raport cu ambele coordonate proiective w şi w^* şi de grad 2 în raport cu fiecare în parte.*

Prin urmare unei suprafeţe R din familia R îi corespund prin plane tangente semiparalele două suprafeţe R^* din familia R^*.

Interpretarea geometrică se poate găsi considerînd înfăşurătoarea planelor tangente T şi T^* utilizate mai sus.

Într-un reper local convenabil ales prin vectorii $\bar{\rho}_u(1,0,0,0)$, $\bar{\alpha}_u(0,1,0,0)$, $\bar{\alpha}_v(0,0,1,0)$ şi $\bar{\alpha}(0,0,0,1)$ obţinem înfăşurătoarea planelor tangente duse la R dealungul generatoarei (g) prin ecuaţiile:

$$\begin{cases} (ax_1 + bx_2 + cx_3)^2 - 4x_1(dx_2 - cx_3) = 0, \\ x_1 - bx_4 + d = 0, \end{cases} \tag{4}$$

care reprezintă un con nedegenerat (coeficienţii a, b, c, d sînt proiecţiile vectorului $\bar{\rho}_v$ în sistemul ales).

Acest con intersectat cu hiperplanul impropriu $x_5 = 0$ determină o conică (C) nedegenerată care este figura improprie corespunzătoare înfăşurătoarei planelor tangente duse familiei R şi este aşezată în planul Π.

Un calcul analog arată că figura improprie corespunzătoare înşurării planelor tangente duse la R^* este o altă conică (C^*) nedegenerată situată în alt plan Π^*.

Hiperplanul impropriu este un S_3, prin urmare intersecţia celor două plane este o dreaptă improprie Δ. Fie t o tangentă dusă într-un punct al conicii (C); ea este figura improprie corespunzătoare planului tangent T dus la o suprafaţă R. Această tangentă t are comun cu dreapta Δ un punct I, de unde se pot duce două tangente t_1 şi t_2 la conica (C). Prin urmare rezultă două plane tangente T_1^* şi T_2^* duse la suprafeţele R_1^* şi R_2^*.

Planele tangente la suprafeţele R_1^* şi R_2^* sînt semiparalele cu planul tangent dus la R, deoarece ele au un punct comun impropriu I.

Astfel se explică geometric rezultatul obţinut fixînd pe w în relaţia (3); atunci corespund două valori w^* care determină două suprafeţe R^*.

De asemenea, prin punctul I trece încă o tangentă t' la conica C; prin urmare mai există o suprafaţă R' ce admite ca suprafeţe, corespunzătoare prin semiparalelismul planelor tangente, suprafeţele R_1^* şi R_2^*.

Prin urmare corespondenţa cercetată este astfel realizată încît unei suprafeţe R îi corespund suprafeţele R_1^* şi R_2^*,

$$R \begin{array}{l} \nearrow R_1^* \\ \searrow R_2^* \end{array}$$

şi există 2 suprafeţe R şi R' ce admit ca suprafeţe corespunzătoare prin semiparalelismul considerat pe R_1^* şi R_2^*.

În cazul particular $\bar{\alpha} = \bar{\alpha}^*$, corespondenţa (3) se reduce la un produs de două omotetii.

Corespondenţa de mai sus poate fi stabilită şi între familii R de pe varietăţi de tip (V_1^2) şi (V_2^2), cînd conica (C) folosită mai sus este degenerată.

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CORRESPONDANCES DANS LE SENS DE O. MAYER ENTRE DES FAMILLES DE VARIÉTÉS R

(Résumé)

D'après O. Mayer [1]...[6] une famille R est formée par ∞^1 courbes situées sur une surface réglée, ainsi que par chaque point d'une génératrice g passe une seule courbe appartenant à la famille R. On généralize ce problème dans un S_n affine, en considérant ∞^1 variétés d'ordre k situées sur une variété réglée (V) de l'ordre $k+1$, ainsi que par chaque point d'une génératrice de (V) passe une variété R de la famille.

Planele tangente la suprafeţele R_1' şi R_2' sînt conjugate cu planul tangent din la R determinate de un punct comun impropriu A.

Astfel se explică geometric rezultatul obţinut fixînd pe x în relaţia (3): ultimei corespund două valori x' care determină două suprafeţe R_1', R_2'.

De asemenea, prin punctul A trece linia o tangentă t' la conica C; prin urmare mai există o suprafaţă R' ce admite ca suprafeţe corespunzătoare prin semiparalelismul planelor tangente, suprafeţele R_1' şi R_2'.

Prin urmare corespondenţa corectată este astfel realizată: fiecărei unei suprafeţe R îi corespund suprafeţele R_1' şi R_2'.

şi există 2 suprafeţe R şi R' ce admit ca suprafeţe corespunzătoare prin semiparalelismul considerat pe R_1' şi R_2'.

In cazul particular $x = x'$, corespondenţa (3) se reduce la un produs de două obiectivităţi.

Corespondenţa de mai sus poate fi stabilită şi între familii R de pe varietăţi de tip (1^2) şi (1^3), cînd conica (C) foloseşte mai sus este degenerată.

Iaşi, la 26 februarie 1975

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CORESPONDANŢA DANS LE SENS DE O. MAYER ENTRE DES FAMILLES DE VARIÉTÉS R

(Résumé)

Dans les O. Mayer [1]...[4] une famille R est formée par ∞^1 courbes situées sur une surface réglée, ainsi que par chaque point d'une génératrice il passe une seule courbe appartenant à la famille R. On généralise le problème dans un S_3 affine, en considérant ∞^1 variétés d'ordre k situées sur une variété réglée (V) de l'ordre $k+1$, ainsi que par chaque point d'une génératrice de (V) passe une variété R de la famille.

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SUR DES CERTAINES STRUCTURES TOPOLOGIQUES DANS UN ANNEAU DE SÉRIES FORMELLES (I)

PAR

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0. Introduction

Nous traitons dans cet article quelques questions concernant certaines topologies dans un anneau de series formelles, qui peuvent être utilisées, en particulier, dans la théorie de certaines systèmes d'équations aux dérivées partielles [4], [5].

Les notions et résultats principaux employés dans ce travail se trouvent dans [1]...[3], [7]...[10].

D'abord nous exposons un certain nombre de résultats algébriques relatifs à un anneau de séries formelles $A[[X_i]]_{i\in I}$ (§ **1**). On définit l'ordre partiel ω_J, $J \subset I$, et l'ordre strict Ω, d'une série formelle. On démontre que toute série formelle vectorielle peut être identifiée avec une série formelle à coefficients vectoriels (Proposition 1). On définit les séries formelles à un nombre arbitraire d'indéterminées. Ensuite nous étudions les topologies des ordres définis ci-dessus. On définit les applications de substitution φ_v, et de composition régulière ψ_u (§ **2**).

Enfin, dans le troisième paragraphe nous cherchons l'image de la composition régulière de deux séries par l'application de dérivation partielle D^r.

La seconde partie de cet étude sera consacrée aux applications de substitution et de composition dans la topologie forte, à le résolution des équations dans un anneau de séries formelles, de même qu'aux applications de substitution et de composition dans la topologie faible.

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1. Séries formelles

1.1. Soit I, un ensemble fini d'indices, Card $(I) = n$, et $\mathbf{I} = N^I$, le monoïde produit. Pour tout $m = (m_i)_{i \in I} \in \mathbf{I}$, et $J \subset I$, notons $m_J = (m'_i)_{i \in I} \in \mathbf{I}$, avec $m'_i = m_i$ si $i \in \mathbf{I}$, et $m'_i = 0$ si $i \notin \mathbf{I}$. Notons $|m|_J = |m_J| = \sum_{i \in I} m_i$. En particulier posons $|m|_I = |m|$ et $m! = \prod_{i \in I} m_i!$.

Pour chaque indice $k \in I$, considérons l'élément $e(k) = (m_i)_{i \in I} \in J$, avec $m_k = 1$ et $m_i = 0$ şi $i \neq k$, $i \in I$.

L'ensemble $\mathbf{I}$ est totalement ordonné par l'ordre lexicographique, noté $\prec$, et defini de la façon suivante : $m \prec m'$ si $|m| < |m'|$ ou $|m| = |m'|$mais il existe un $j \in I$ tel que $m_j > m'_j$ et $m_i = m'_i$ pour tout $i < j$, $i \in I$.

Lorsque I est un ensemble quelconque considérons la partie stable de $\mathbf{I}$, notée $(\mathbf{I}) = N^{(I)}$, formée des éléments $m = (m_i)_{i \in I}$ pour lesquelles $m_i = 0$, sauf pour un nombre fini d'inices $i \in I$. Elle est identique à $\mathbf{I}$ si I est fini. Le monoïde additif $(\mathbf{I})$ satisfait la condition :

(P) *Pour tout élément* $m \in (\mathbf{I})$, *il n'y a qu'un nombre fini de couples* $(m', m'') \in (\mathbf{I}) \times (\mathbf{I})$ tels que $m' + m'' = m$.

1.2. Soit A un anneau commutatif, possedant un élément unité (noté 1), intègre.

L'ensemble produit $F = A^{\mathbf{I}}$, muni d'une structure algébrique d'anneau, avec l'addition usuelle et la multiplication de Cauchy, est par définition *l'anneau de series formelles* par rapport aux n indéterminées à coefficients dans A [1], [8], [9]. Lorsque $I = \emptyset$ posons par convention $F = A$.

Pour tout $m \in \mathbf{I}$, notons par $p_m : F \to A$, et $i_m : A \to F$, les projections et les injections canoniques du produit F. C'est-à-dire pour tout $u = (u_m)_{m \in \mathbf{I}} \in F$ nous avons $p_m(u) = u_m$, $m \in \mathbf{I}$; et pour tout $a \in A$, $i_m(a) = (u_{m'})_{m' \in \mathbf{I}}$, avec $u_m = a$ et $u_{m'} = 0$ pour tout $m' \neq m$, $m' \in \mathbf{I}$. Les séries formelles $i_{e(k)}(1) = X_k$, $k \in I$, s'appellent les indéterminées de F. Pour tout $m \in \mathbf{I}$, nous avons

$$i_m(1) = \prod_{i \in I} X_i^{m_i} = X^m.$$

Utilisons les notations consacrées, $F = A[[X_i]]_{i \in I}$ pour l'anneau des séries formelles, et $F' = A[X_i]_{i \in I}$ pour le sous-anneau des polynomes.

Soient $J \subset I$, $L = I - J$. Alors l'anneau $A[[X_i]]_{i \in I}$ est isomorphe à l'anneau des séries formelles $A[[X_l]]_{l \in L}[[X_j]]_{j \in J}$.

1.3. *L'ordre d'une série formelle* $u \in A[[X_i]]_{i \in I}$, est le nombre naturel $\omega(u) = \min\{|m| ;\ m \in \mathbf{I},\ p_m(u) \neq 0\}$, si $u \neq 0$ et $\omega(u) = +\infty$, si $u = 0$.

L'ordre partiel d'une serie formelle $u \in A[[X_i]]_{i\in I}$, par rapport à un sous-ensemble d'indéterminées $(X_j)_{j\in J}, J \subset I$, est l'ordre de $u \in A[[X_l]]_{l\in L}[[X_j]]_{j\in J}$. Il est noté par $\omega_J(u)$. Lorsque $J = \{k\}$, notons $\omega_J(u) = \omega_k(u)$.

L'ordre strict de $u \in A[[X_i]]_{i\in I}$, noté $\Omega(u)$, est défini par $\Omega(u) = m$, $m=(m_i)_{i\in I} \in \mathbf{I}$, si $\omega_i(u) = m_i$, $i \in I$.

Pour tout $J \subset I$ et $k \in J$, nous avons

$$\omega_k(u) \leqslant |\Omega_J(u)| \leqslant \omega_J(u) \leqslant \omega(u).$$

Remarquons que n'importe lequel de ces ordre a les propriétés suivantes :

$$\varphi(u\,v) = \varphi(u) + \varphi(v), \quad \varphi(u+v) \leqslant \inf\{\varphi(u), \varphi(v)\},$$

$$\varphi(0) = +\infty, \quad \varphi(1) = 0,$$

pour tout $u, v \in A[[X_i]]_{i\in I}$.

Notons par $\deg_J(u)$, le degré de $u \in A[[X_i]]_{i\in I}$, élément de l'anneau $A[[X_l]]_{l\in L}\,[X_j]_{j\in J}$. On peut définir d'une façon analogue le degré d'indice k, $k \in I$, et le degré strict, noté Deg (u). On considère par convention $\deg(0) = -\infty$.

1.4. Lorsque Γ est un ensemble quelconque, alors A^Γ est l'anneau produit, avec les lois de composition usuelles.

P r o p o s i t i o n 1. *L'anneau* $(A[[X_i]]_{i\in I})^\Gamma$ *est isomorphe à l'anneau* $A^\Gamma\,[[X_i]]_{i\in I}$.

En effet, il existe une application biunivoque de $(A^{\mathbf{I}})^\Gamma$ sur $(A^\Gamma)^{\mathbf{I}}$ définie par :

$$((u_m^\gamma)_{m\in\mathbf{I}})_{\gamma\in\Gamma} \longleftrightarrow ((u_m^\gamma)_{\gamma\in\Gamma})_{m\in\mathbf{I}},$$

qui est évidemment un isomorphisme du premier de ces anneau sur le second.

Tout élément de l'anneau $(A[[X_i]]_{i\in I})^\Gamma$ est appellé *série formelle vectorielle.*

Il résulte de la proposition 1 que tout anneau de séries formelles vectorielles peut être identifié avec un anneau de séries formelles à coefficients vectoriels.

L'ordre d'une serie formelle vectorielle $U = (u_\gamma)_{\gamma\in\Gamma}, u_\gamma \in A[[X_i]]_{i\in\mathbf{I}}$ $\gamma \in \Gamma$, est défini par $\omega(U) = \inf\limits_{\gamma\in\Gamma} \omega(u_\gamma)$ et est égal à l'ordre de la série formelle correspondante de l'anneau $A^\Gamma\,[[X_i]]_{i\in I}$. On peut définir de la même façon, l'ordre partiel et l'ordre strict d'une série formelle vectorielle.

1.5. *La dérivation partielle d'ordre* r, $r \in \mathbf{I}$, dans $A[[X_i]]_{i\in I}$ est l'application :

$$D^r : A[[X_i]]_{i\in I} \to A[[X_i]]_{i\in I},$$

définie par :

$$p_m \cdot D^r = \frac{(m+r)!}{m!} p_{m+r}, \; m \in \mathbf{I}.$$

De la définition il résulte aussitôt que :

$$\operatorname{Ker}(D^r) = \{u \,;\, u \in A[[X_i]]_{i\in I}, \exists\, i \in I, \deg_{\infty}(u) < r_i\}$$

et que pour tout $v \in A[[X_i]]_{i\in I}$, il existe une série formelle et une seule $u \in A[[X_i]]_{i\in I}$, telle que:

$$\Omega(u) = r, \qquad D^r u = v.$$

1.6. Lorsque I est un ensemble quelconque, alors une série formelle aux indéterminnées $(X_i)_{i\in I}$, à coefficients dans A, est tout élément $u = (u_m)_{m\in\mathbf{I}}$ de l'anneau $A^{(\mathbf{I})}$ pour lequel $u_m = 0$ pour tout $m \in (\mathbf{I}), |m| = k$, $k \in N$, sauf pour un nombre fini d'indices. Notons par (F) leur ensemble. Remarquons que de la condition (P) il résulte que le produit de Cauchy est défini.

Dans ce cas toute série formelle $u \in A[[X_i]]_{i\in I}$, s'écrit d'une seule manière $u = ((u_m)_{m\in\mathbf{I}(k)})_{k\in N}$, où m parcourt l'ensemble $\mathbf{I}(k) = \{m \,;\, m \in (\mathbf{I}), |m| = k\}$, $k \in N$; et donc $A[[X_i]]_{i\in I} = \prod_{k\in N} \coprod_{m\in\mathbf{I}(k)} A_m$, $A_m = A$, $m \in (\mathbf{I})$ (le produit de sommes directes).

Remarquons que toutes les notions et les résultats antérieurs sont valable aussi dans ce cas.

Soient I et J deux ensembles équipotents, φ une application biunivoque de I sur J. Alors l'application de $A[[X_i]]_{i\in I}$ sur $A[[X_j]]_{j\in J}$, qui à tout élément $(u_m)_{m\in J} \in A[[X_i]]_{i\in I}$, fait corespondre l'élément $(u_{\varphi(m)})_{m\in\mathbf{I}} \in A[[X_j]]_{j\in J}$, où pour tout $m=(m_i)_{i\in I} \in \mathbf{I}$, $\varphi(m) = (m_{\varphi(i)})_{i\in I} \in \mathbf{I}$, est un isomorphisme du premier de ces anneaux sur le second.

2. Topologies fortes de $A[[X_i]]_{i\in I}$

2.1. Soit

$$\mathcal{M}(\omega_J) = \{u \in A[[X_i]]_{i\in I}, \omega_J(u) \geqslant 1\}, \quad J \subset I.$$

Des propriétés de l'ordre partiel ω_J, il résulte que l'ensemble $\mathcal{M}(\omega_J)$ est un idéal bilatéral dans $A[[X_i]]_{i\in I}$ et

$$[\mathcal{M}(\omega_J)]^k = \{u \in A[[X_i]]_{i\in I}, \omega_J(u) \geqslant k\}, \quad k \in N.$$

Les ensembles $\mathcal{M}[(\omega_J)]^k$, $k \in N$, forment un système fondamental de voisinages de 0, dans une topologie sur $A[[X_i]]_{i\in I}$, qui s'appelle *la topologie de l'ordre* ω_J, et est designée par $T(\omega_J)$. Lorsque I parcourt l'ensemble $P(I)$, les topologies $T(\omega_J)$ sont les topologies fortes de $A[[X_i]]_{\in I}$, Chacune de ces topologies est métrisable et l'anneau topologique $A[[X_i]]_{i\in I}$. est complet (les démonstrations sont semblables à celles données dans [8]...[10] pour $T(\omega)$.

L'anneau $A[[X_i]]_{i\in I}$ devient un anneau valué non-archimédien, en posant :

$$|u| = r^{\omega_J(u)}, \quad r \in R, \quad 0 < r < 1.$$

Comme pour tout $J_1 \subset I$, $J_1 \cap J = \emptyset$, nous avons :

$$A[[X_j]]_{j\in J_1} \cap (\mathcal{M}(\omega_J))^k = \{0\}, \quad k \geqslant 1,$$

il est clair que la topologie $T(\omega_J)$ induit sur $A[[X_j]]_{j\in J_1}$ la topologie discrète. Mais, pour tout $J_1 \subset I$, $J_1 \cap J \neq \emptyset$, elle induit sur $A[[X\ ]]_{j\in J_1}$, la topologie de l'ordre $\omega_{J\cap J_1}$.

Pour tout J_1, $J_2 \subset J$, $J_1 \subset J_2$, et tout $j \in J_1$, nous avons :

$$T(\omega\ ) \prec T(\omega_{J_1}) \prec T(\omega_{J_2}) \prec T(\omega).$$

En effet, $\omega_{J_1}(u) \leqslant \omega_{J_2}(u)$, et donc pour tout $k \geqslant 1$:

$$(\mathcal{M}(\omega_{J_1}))^k \subset (\mathcal{M}(\omega_{J_2}))^k,$$

c'est-à-dire l'application :

$$1_F : (F\ ;\ T(\omega_{J_2})) \to (F;\ T(\omega_J\ ))$$

est continue et par suite : $T(\omega_{J_1}) \prec T(\omega_{J_2})$.

Il résulte aussi tôt de la définition que l'application $D^r : F \to F$, $r \in \mathbf{I}$, est continue, quand F est muni de la topologie $T(\omega_J)$.

2.2. Proposition 2. *La série ayant le terme général* $u^{(k)} \in A[[X_i]]_{i\in \mathbf{I}}$ $k \in N$, *est convergente dans* $T(\omega_J)$, $J \supset I$, *si et seulement si, la suite* $\{u^{(k)}\}_{k\in N}$ *est convergente vers zéro dans* $T(\omega_J)$.

En effet, soit $v^{(k)} = \sum_{i \leqslant k} u^{(i)}$, $k \in N$. Supposons que $\lim u^{(i)} = 0$ dans $T(\omega_J)$, c'est-à-dire, $\lim \omega_J\,(u^{(i)}) = +\infty$. Il suit que :

$$\omega_J\,(v^{(i+j)} - v^{(i)}) = \omega_J(u^{(i+1)} + \ldots + u^{(i+j)}) \geqslant \inf_{I \leqslant k \leqslant j} \{\omega_{\mathbf{I}}(u^{(i+k)})\} \geqslant k$$

pour tout $i \geqslant i_0(k)$, et tout $j \in N$.

Il résulte que la suite $\{v^k\}_{k\in N}$ est une suite de Cauchy dans $T(\omega_J)$, donc convergente, parce que $(A[[X_i]]_{i\in I}, T(\omega_J))$ est un anneau topologique complet.

La réciproque est évidente.

Remarque. Soient $u = (u_m)_{m\in \mathbf{I}} \in A[[X_i]]_{i\in I}$ et $V = (v_i)_{i\in I}$, $v_i \in A[[Y_j]]_{j\in J}$, $i \in I$, avec $\omega_{J_1}(V) \geqslant 1$, $J_1 \subset J$. Alors $\omega_{J_1}(u_m V^m) \geqslant |m|_{J_1}$, et donc de la proposition 2 il résulte que la série avec le terme général $u_m V^m$, $m \in \mathbf{I}$, considérée comme série simple par la relation d'ordre canonique de $\mathbf{I}$, est convergente dans $(A[[Y_j]]_{j\in J}, T(\omega_{J_1}))$. Par définition sa somme est noté par $u(V)$ et s'appelle *la composition ou la substitution régulière* des séries formelles u et V. En particulier, quand $v_i = X_i$, $i \in I$, nous avons

$$u(x) = \sum_{m\in \mathbf{I}} u_m X^m \quad \text{et} \quad u_m = \frac{1}{m!} D^m u(0), \quad m \in \mathbf{I}.$$

Lorsque $u(X)$ est un polynome, la composition de $u(X)$ et $V(Y)$ est régulière pour toute série formelle vectorielle $V(Y)$.

L'application

$$\varphi_V : A[[X_i]]_{i\in I} \to A[[Y_j]]_{j\in J}$$

définie par $u(X) \to u(V(Y))$, pour tout $V(Y) \in (A[[Y_j]]_{j\in J})^I$ fixée, avec la condition $\omega_{J_1}(V) \geqslant 1$, $J_1 \subset J$, et tout $u(X) \in A[[X_i]]_{i\in I}$, est appellée *application de substitution régulière.*

L'application

$$\psi_u : (A[[Y_j]]_{j\in J})^I \to A[[Y_j]]_{j\in J}$$

définie par $V(Y) \to u(V(Y))$, pour tout $u(X) \in A[[X_i]]_{i\in I}$ fixée, et tout $V(Y) \in (A[[Y_j]]_{j\in J})^I$, $\omega_{J_1}(V) \geqslant 1$, $J_1 \subset J$, est appellée *application de composition régulière.*

2.3. Soit maintenant I un ensemble d'indices quelconques. Alors dans l'anneau $A^{(\mathbf{I})}$, considérons la topologie T, intersection des topologies $T(\omega_J)$, lorsque J parcourt la famille des parties finies de I. On l'appelle *la topologie forte de* $A[[X_i]]_{i\in I}$. Toute série formelle $u \in A[[X_i]]_{i\in I}$ s'écrit d'une seule manière:

$$u(X) = \sum_{k\in N} \sum_{m\in \mathbf{I}(k)} u_m X^m = \sum_{m\in (\mathbf{I})} u_m X^m,$$

qui est une série convergente dans T.

Lorsque I, J sont deux ensemble quelconques, les applications de substitution et de composition régulière peuvent être définies de la même manière.

Lorsque $\Omega(u) \neq 0$, alors il existe une seule $v \in A[[X_i]]_{i\in I}$ telle que $\Omega(v) = 0$ et $u(X) = X^{\Omega(u)}\, v(X)$.

3. La dérivée de la composition des séries formelles

3.1. D'abord quelques *notations*. Pour tout $m = (m_i)_{i\in I} \in \mathbf{I}$, notons par $\{m(j)\}_{1\leqslant j\leqslant |m|}$ la suite formée des nombres 1 de m_1 fois, 2 de m_2 fois... et n de m_n fois. Notons par M, l'ensemble de toutes les suites distinctes qui peuvent être obtennes de cette suite par des permutations. Notons par $\overline{m}$ un élément quelconque de M. Par exemple si $m = (1, 2, 0)$, alors la suite initiale est $\{1, 2, 2\}$ et $M = \{\{1, 2, 2\}, \{2, 1, 2\}, \{2, 2, 1\}\}$.

Pour tout $r = (r_j)_{j\in J} \in \mathbf{I}$, notons par $\widetilde{r}$ le multi-indice formé de p groupes de composantes, tout groupe formé de r_j éléments égaux à $1, j\in J$, si $r_j \neq 0$, et un zéro si $r_j = 0$. Pour tout $q \in N$, $q \leqslant |r|$, considérons l'équation : $\sum\limits_{1\leqslant j\leqslant q} \widetilde{s}(j) = \widetilde{r}$ (**I**). Notons par $\widetilde{S}_q$ les q - ouples de solutions distinctes de cette équation. Comme plus haut pour tout $\widetilde{s}(j) \in \widetilde{S}_q$, il exite un $s(j) \in \mathbf{I}$ tel que la composante d'indice k de $s(j)$ est la somme des composantes du groupe k de $\widetilde{s}(j)$. Notons par S_q l'ensemble des q-ouples de multi-indices obtenues de cette façon. Notons par $\widetilde{s}$ un élément quelconque de cet ensemble. Par exemple si $r = (2, 1, 0)$, alors $\widetilde{r} = (1, 1 ; 1 ; 0)$. Supposons que $q = 2$. Considerons l'équation

$$\widetilde{s}(1) + \widetilde{s}(2) = (1, 1 ; 1 ; 0).$$

Les solutions de cette équation sont les couples : $((1, 1 ; 0 ; 0), (0, 0 ; 1 ;0))$ $((1, 0 ; 1 ; 0), (0, 1 ; 0 ; 0))$ et $((0, 1 ; 1 ; 0), (1, 0 ; 0 ; 0))$. Alors :

$$S_2 = ((2, 0, 0), (0, 1, 0)), ((1, 1, 0), (1, 0, 0)), ((1, 1, 0), (1, 0, 0)).$$

Remarquons que pour tout $\overline{s} = (s(j))_{1\leqslant j\leqslant q}$, nous avons $\sum\limits_{1\leqslant j\leqslant q} s(j) = r$.

3.2. Proposition 3. *Soient* $u(X) \in A[[X_i]]_{i\in I}$ *et* $V(Y) = (v_i(Y))$, $v_i(Y) \in A[[Y_j]]_{j\in J}$, $i \in I$, *avec* $\omega_{J_1} \geqslant 1$, $J_1 \subset J$. *Alors*

$$D_Y^r u(V(Y)) = \sum_{\substack{m\in \mathbf{I} \\ 1\leqslant |m|\leqslant |r|}} D_X^m u(V) \sum_{\overline{m}\in M} \sum_{\overline{s}\in S_{|m|}} \prod_{1\leqslant j\leqslant |m|} D_Y^{s(j)} v_{m(j)}, \tag{3.1}$$

avec les notations introduites.

La proposition se démontre par réccurence sur r, $r \in J$.

Remarquons d'abord que pour tout $q \in N$ et tout $k \in J$, nous avons:

$$D_Y^{e(k)}[v(Y)]^q = q[v(Y)]^{q-1} D_Y^{e(k)} v(Y), \tag{3.2}$$

relation qui on peut démontrer aussitôt par réccurence sur q.

Soit

$$u(X) = \sum_{m \in \mathbf{I}} u_m X^m.$$

Alors

$$u(V(Y)) = \sum_{(m_i) \in \mathbf{I}} (u_{m_i}) \prod_{i \in I} v_i(Y)^{m_i}.$$

Soit $r = e(k)$, $k \in J$. De la continuité de l'application D_Y^r dans la topologie $T(\omega_{J_1})$, il résulte

$$D_Y^{e(k)} u(V(Y)) = \sum_{(m_i) \in \mathbf{I}} u_{(m_i)} D_Y^{e(k)} \left(\prod_{i \in I} [v_i(Y)]^{m_i} \right) =$$

$$= \sum_{(m_i) \in \mathbf{I}} u_{(m_i)} \sum_{i \in I} m_i [v_1(Y)]^{m_1} \dots [v_i(Y)]^{m_i - 1} \dots [v_n(Y)]^{m_n} D_Y^{e(k)} v_i(Y) =$$

$$= \sum_{i \in I} D_Y^{e(k)} v_i(Y) \sum_{(m_i) \in \mathbf{I}} m_i u_{(m_i)} [v_1(Y)]^{m_1} \dots [v_1(Y)]^{m_i - 1} \dots [v_n(Y)]^{m_n} =$$

$$= \sum_{i \in I} (D_X^{e'(i)} u(V))(D_Y^{e(k)} v_i(Y)).$$

C'est-à-dire pour tout $r \in \mathbf{I}$, $|r| = 1$, nous avons:

$$D_Y^r u(V(Y)) = \sum_{|m|=1} (D_X^m u(V)) (D_Y^r v_{\overline{m}}(Y)), \tag{3.3}$$

où $m = e'(k)$ et $\overline{m} = k$, donc la relation (1).

Supposons maintenant que la relation (1) est vraie pour tout $r \in J$, $|r| \leqslant p$, $p \in N$ fixé. Alors pour tout $r \in \mathbf{I}$, $|r| = p$ et tout $k \in J$, de l'hypothèse inductive il résulte :

$$D_Y^{e(k)} u(V(Y)) = D_Y^{e(k)}(D_Y^r u(V(Y)) =$$

$$= \sum_{\substack{1 \leqslant |m| \leqslant |r| \\ m \in \mathbf{I}}} D_Y^{e(k)} (D_X^m u(V)) \sum_{\overline{m} \in M} \sum_{s \in S_{|m|}} \prod_{1 \leqslant j \leqslant |m|} D_Y^{s(j)} v_{m(j)}(Y) +$$

$$+ \sum_{\substack{1 \leqslant |m| \leqslant |r| \\ m \in \mathbf{I}}} D_X^m u(V) \sum_{\overline{m} \in M} \sum_{s \in S_{|m|}} D_Y^{e(k)} \left(\prod_{1 \leqslant j \leqslant |m|} D_Y^{s(j)} v_{m(j)}(Y) \right).$$

Mais de la relation (3) il résulte que :

$$D_Y^{r+e(k)} u(V(Y)) = \sum_{\substack{1 \leqslant |m| \leqslant |r| \\ m \in J \\ i \in \mathbf{I}}} (D_X^{m+e'(i)} u(V)) (D_Y^{e(k)} v_i(Y)) \sum_{\substack{\bar{m} \in M \\ \bar{s} \in S_{|m|}}} \prod_{1 \leqslant j \leqslant |m|} (D_Y^{s(j)} v_{m(j)}) +$$

$$+ \sum_{\substack{1 \leqslant |m| \leqslant |r| \\ m \in \mathbf{I}}} D_X^m u(V) \sum_{\substack{\bar{m} \in M \\ \bar{s} \in S_{|m|}}} \sum_{1 \leqslant j \leqslant |m|} (D_Y^{s(1)} v_{m(1)}) \dots (D_Y^{s(j)+e(k)} v_{m(j)}) \dots (D_Y^{s(m)} v_{m(|m|)}).$$

Notons $m + e'(i) = m'$, et remarquons que les m' - ouples, $(e(k), s(1), \dots, s(|m|))$ et $(s(1), \dots, s(j) + e(k), \dots, s(|m'|))$ parcourent l'ensemble $S_{|m'|}$, et $(i, m(1), \dots, m(|m|))$, $(m(1), \dots, m(j), \dots, m(|m'|))$ l'ensemble M' correspondant. Il résulte aussitôt la relation (3.1) par un réarrengement des termes d'apres les ordres de dérivation de $u(V)$.

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ASUPRA UNOR STRUCTURI TOPOLOGICE ÎNTR-UN INEL DE SERII FORMALE (I)

(Rezumat)

Se studiază o serie de probleme privitoare la structurile topologice, într-un inel de serii formale. Sînt definite noţiunile de ordin parţial şi de ordin strict şi topologiile corespunzătoare şi se studiază imaginea compoziţiei a două serii formale prin aplicaţia de derivare parţială (propoziţia 3).

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D. C. 519.5 ; 517.

DEPENDENCE ON INITIAL CONDITIONS OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION WITH A SET MAPPING AS UNKNOWN

BY

DAN BORŞ

Let us consider the differential equation [1]

$$\varphi'(x) = f(x, \varphi(A_x)), \tag{1}$$

where φ' is the derivative of the mapping f, x is an arbitrary element of an open subset X of a metric space M, f is a continuous and bounded mapping which satisfies the Lipschitz condition in the second argument with the constant L and which applies an open subset D of $M \times R^1$ into the set R^1 of real numbers, φ is a countably additive set mapping defined on a tribe T of subsets of X which satisfies the property 1.3 of [1], A_x is an element of a subset T' of T such that there exists a bijection $F : X \to T'$ which satisfies the property 1.4 of [1].

Let U be the complete metric space of the continuous and bounded mappings $u : X \to R^1$, with the metric

$$\delta(u, v) = \sup_{x \in X} | u(x) - v(x) | .$$

If $L \cdot \mu(X) < 1$, $\mu : T \to R^1$ being a positive measure, there exists a unique mapping $\varphi : T \to R^1$ such that its restriction to T' satisfies (1) and has arbitrary given values in any intersection of the sets of T' with a given zero measure set A_0.

The graph of a solution $\varphi : T' \to R^1$ of the differential equation (1) is a subset of the quasivector space $T \times R^1$, which elements are of the form $(A_x, \varphi(A_x))$ for any x in X ; this graph is uniquely determined, under the above conditions, and it contains the elements of the nonbanal quasivector subspace $(T \cap A_0) \times R^1$ of $T \times R^1$, which belong to the sub-

set $(T' \cap A_0) \times R^1$. In this case we say that the solution φ passes through the set $(T' \cap A_0) \times R^1$ and it is unique if we know its values in the elements of the set $T' \cap A_0$.

We call the set $T' \cap A_0$ the initial set; a mapping $u_0 \in U$ which values define the values of the solution φ in the initial set $T' \cap A_0$, that is

$$\varphi(A_x \cap A_0) = u_0(x), \quad x \in X, \tag{2}$$

we call initial mapping.

We call the relations (2), for any $x \in X$ and $A_x \in T'$, the initial conditions of the solution φ for a given A_0.

We replace the notation $\varphi(A_x) = u(x)$, $x \in X$, used in [1], by a more suggestive one

$$\varphi(A_x, A_0, u_0) = u(x, A_0, u_0), \quad x \in X.$$

P r o p o s i t i o n. *The solution φ depends continuously on initial conditions* (2), *that is on the initial mapping u_0.*

P r o o f. Let u_0, u_0' be two initial mappings which define the initial conditions

$$\varphi(A_x \cap A_0) = u_0(x), \quad \varphi(A_x \cap A_0') = u_0'(x) \tag{2'}$$

for any $x \in X$ and two zero measure fixed sets A_0 and A_0'.

The solutions $\varphi(A_x, A_0, u_0) = u(x, A_0, u_0)$ and $\varphi(A_x, A_0', u_0') = u(x, A_0', u_0')$ of the differential equation (1) are uniquely determined under the initial conditions (2′); they are fixed elements of the contractions $\Phi : U \to U$ and $\Psi : U \to U$ defined by

$$(\Phi u)(x) = u_0(x) + \int_{A_x} f(y, u(y))\, d\mu,$$

$$(\Psi u)(x) = u_0'(x) + \int_{A_x'} f(y, u(y))\, d\mu.$$

Let $\varepsilon > 0$ be an arbitrary real number such that $\delta(u_0, u_0') < \varepsilon$; then

$$\delta(\Phi u, \Psi u) = \sup_{x \in X} |(\Phi u)(x) - (\Psi u)(x)| =$$
$$= \sup_{x \in X} |u_0(x) - u_0'(x)| = \delta(u_0, u_0') < \varepsilon.$$

By [2, p. 177], it follows

$$\delta(u(x, A_0, u_0), \; u(x, A_0', u_0')) < \varepsilon\,(1 - L.\,\mu(X))^{-1}$$

that is, $u = u(x, A_0, u_0)$ is a continuous mapping on u_0. Q.E.D.

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DEPENDENŢA DE DATELE INIŢIALE A SOLUŢIILOR UNEI ECUAŢII DIFERENŢIALE CU O FUNCŢIE DE MULŢIME CA NECUNOSCUTĂ

(Rezumat)

Se stabileşte dependenţa continuă a soluţiilor ecuaţiei diferenţiale (1) de condiţiile iniţiale (2).

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TWO REMARKABLE EXTENSIONS OF THE LEIBNIZ DIFFERENTIATION FORMULA

BY

H. W. GOULD

1. Introduction

The Leibniz differentiation formula

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k u \, D^{n-k} v, \quad D = \frac{d}{dx}, \tag{1.1}$$

is certainly well-known. However it does not seem at all widely known that the formula may be generalized in at least two slightly different ways as follows:

$$\sum_{k=0}^{n} \binom{n}{k} D^{k-1}\{[f(x)]^k u'(x)\} D^{n-k-1}\{[f(x)]^{n-k} v'(x)\} = D^{n-1}\{[f(x)]^n D(uv)\}, \quad u' = Du, \tag{1.2}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} D^{k-1}\{[f(x)]^k u'(x)\} D^{n-k}\{[f(x)]^{n-k} v(x)\} = D^n\{[f(x)]^n uv\}, \tag{1.3}$$

where $D^{-1}u' = u$. Formula (1.2) was given by Cauchy [3, p. 72, eq. (35)]. He made use of it there to obtain a formula of Abel [1]. Sophus Lie in Vol. 2, p. 294 of the 1 881 edition of Abel's collected works [2] remarks that Abel's formula is a special case of a formula given by Cauchy in his *Exercises de Mathématiques*. Connections between Abel's formula and other binomial identities may be found traced in [5]...[7], [12]. Formula (1.3) seems to be due to Walker [14]. In the older literature one finds various references to these formulas. For example, Nekrasov [11, pp. 20, 69—77] makes extensive use of (1.2) but he quotes it from Serret [13]. Crofton [4] obtains Walker's formula by symbolic methods. Walker proved his formula by induction. Such a proof is not exactly trivial because of the appearance of the k-th power of $f(x)$.

The present paper grew from attempts to find a really simple inductive proof. Along the way it was found that the formulas do not appear in

current literature, let alone the fact that they are immediate consequences, as shown below, of L a g r a n g e's extension of the T a y l o r series.

It was found that to every formula like (1.2) or (1.3) there coresponds an analogous formula involving $\binom{a+bk}{k}$ as well as one involving $a(a+bk)^{k-1}/k!$ so that the work sheds some light on the study of combinatorial identities. In particular, formulas (1.2) and (1.3) may be said to be symmetrical and non-symmetrical varieties, respectively, of the possible formulas. A natural companion to these relations turns out in the form (symmetrical)

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mathrm{D}^k\{[f(x)]^k uF\}\,\mathrm{D}^{n-k}\left\{\frac{[f(x)]^{n-k}}{F}\right\} &= \\ = \sum_{k=0}^{n}\binom{n}{k}\mathrm{D}^k\{[f(x)]^k u\}\,\mathrm{D}^{n-k}\{[f(x)]^{n-k}v\}.\end{aligned} \tag{1.4}$$

This relation is the analogue of the combinatorial identity

$$\sum_{k=0}^{n}\binom{a+bk}{k}\binom{c+(n-k)b}{n-k}=\sum_{k=0}^{n}\binom{a+d+bk}{k}\binom{c-d+(n-k)b}{n-k}, \tag{1.5}$$

which is relation (12) in [5]. What is more, (1.4) is the key to the sought-after inductive proof we outline below for (1.3).

Finally we wish to observe that since many widely-studied special functions are definable by a R o d r i g u e s formula of form $\mathrm{D}^n(f^n u)$ it comes as no surprise that such expressions can be handled by the formulas studied in this paper, and we hope in some future account to present a few novel consequences of this fact.

2. Lagrange's Formulas

Almost precisely 200 years ago, L a g r a n g e [9] gave a remarkable paper to the Berlin Academy in which he was able to invert infinite series in a new manner, hoping thereby to find new ways to solve algebraic equations. What he gave is an extension of the T a y l o r series which we may state in the following form. Let $z = x + tf(z)$. Then formally the function $u = u(x)$ satisfies

$$u(z) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{D}^{n-1}\{[f(x)]^n\,\mathrm{D}u(x)\}. \tag{2.1}$$

Whether the series actually converges to u has been studied in detail, and the reader may consult W h i t t a k e r and W a t s o n [15] or similar works on complex analysis. We treat (2.1) in a purely formal manner, using the algebra of formal power series which can be used to justify the use of generating functions to prove finite identities by comparison of coefficients.

We note that L a g r a n g e 's paper is also available in a German translation by M i c h e l s e n [10]. Among the special cases treated by L a g r a n g e we find the formula (converted to modern notation)

$$z^a = \sum_{n=0}^{\infty} \frac{a}{a+bn} \binom{a+bn}{n} t^n, \quad tz^b = z - 1. \tag{2.2}$$

He studied this as an example of how a trinomial equation can be solved for a power of a root. This formula, as noted in [5]...[7] is the genesis of a combinatorial formula found in 1793 by H. A. R o t h e, which says that

$$\sum_{k=0}^{n} (p+qk)\, A_k(a, b)\, A_{n-k}(c, b) = \frac{p(a+c)+qna}{a+c} A_n(a+c, b), \tag{2.3}$$

where

$$A_k(a, b) = \frac{a}{a+bk} \binom{a+bk}{k}.$$

This formula contains a symmetrical case (analogue of (1.2)) when $p = 1$ and $q = 0$, and a non-symmetrical case (analogue of (1.3)) when $p = a$ and $q = b$. What we mean by non-symmetrical is that the formula

$$\sum_{k=0}^{n} A_k(a, b) G_{n-k}(c, b) = G_n(a+c, b), \quad G_k(a, b) = \binom{a+bk}{k},$$

is unlike the case $p = 1$, $q = 0$ of (2.3) in that the series is not symmetrical when we interchange a and c because $A \neq G$. This is not to say that the function is not symmetrical in a and c, which is true.

Now, since $z = x + tf(z)$, if we hold t fixed and differentiate, we find $D_x z = 1/D_z x$ and $D_z x = 1 - tf'(z)$, so that (2.1) can be made to yield the other known form of L a g r a n g e 's expansion

$$\frac{u(z)}{1 - tf'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \{[f(x)]^n u(x)\}. \tag{2.4}$$

This in turn yields the formula

$$\frac{z^a}{(1-b)z + b} = \sum_{n=0}^{\infty} \binom{a+bn}{n} t^n, \quad tz^b = z - 1, \tag{2.5}$$

as a companion to (2.2). This formula is the genesis for (1.5) above.

By choosing appropriate functions, or by noting merely that

$$\lim_{h \to 0} \binom{(a+bk)/h}{k} h^k = \frac{(a+bk)^k}{k!}, \tag{2.6}$$

one may easily find the corresponding A b e l - type formulas in the forms

$$z^a = \sum_{n=0}^{\infty} a \frac{(a+bn)^{n-1}}{n!} t^n, \quad tz^b = \log z, \tag{2.7}$$

and

$$\frac{z}{1 - b \log z} = \sum_{n=0}^{\infty} \frac{(a+bn)^n}{n!} t^n. \tag{2.8}$$

It is from these two formulas that numerous formulas used today in graph theory and statistics may be derived. The reader is referred to Riordan [12] for an extensive account of Abel and other classes of identities.

If we equate coefficients of powers of t in the identity $z^a . z^c = z^{a+c}$ using (2.2) (or (2.7)) we obtain the symmetrical Rothe (or Abel) formula. If we use the identity $z^a . z^c g(z) = z^{a+c} g(z)$ and apply (2.2) and (2.5) (or (2.7) and (2.8)) we obtain the non-symmetrical Rothe (or Abel) formula. Precisely the same reasoning applies as we next turn to the corresponding differentiation formulas.

3. Proofs of (1.2) and (1.3) by Lagrange Series

Expand u and v by (2.1) and multiply the series together. This gives us

$$uv = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^{k-1} [f^k \mathrm{D}u] \, \mathrm{D}^{n-k-1} [f^{n-k} \mathrm{D}v].$$

But (2.1) gives also

$$uv = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{D}^{n-1} [f^n D(uv)],$$

whence by comparison of coefficients of powers of t we have proved (1.2).

To obtain (1.3) we expand u by means of (2.1) and v by means of (2.4), so that we have

$$\frac{uv}{1 - tf'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^{k-1} [f^k \mathrm{D}u] \mathrm{D}^{n-k} [f^{n-k} v] =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{D}^n [f^n uv],$$

by using (2.4) alone, whence we have the desired result.

If we try to find a simple formula by using (2.4) alone we run into a complication. We may expand $u/(1 - tf'(z))$ and $v/(1 - tf'(z))$ using(2.4), but to handle the product $uv/(1 - tf'(z))^2$ requires a more complicated series than (2.1) or (2.4) and the result is not so simple. However, we can consider one easy result: Expand $uF/(1 - tf'(z))$ and $v/F(1 - tf'(z))$ individually by (2.4) and also expand the product which is $uv/(1 - tf'(z))^2$. The result is formula (1.4).

4. Short Inductive Proof of Walker's Formula using (1.4)

Suppose that (1.3) is true for a fixed n. Then, on the one hand,

$$D^{n+1}[uvf^{n+1}]=D\{D^n[uvff^n]\}=D\sum_{k=0}^{n}\binom{n}{k}D^{k-1}[u'f^k]\,D^{n-k}[vff^{n-k}],$$

whence

$$\begin{aligned}D^{n+1}[uvf^{n+1}]&=\sum_{k=0}^{n}\binom{n}{k}D^k[u'f^k]\,D^{n-k}[vf^{n+1-k}]+\\&+\sum_{k=0}^{n}\binom{n}{k}D^{k-1}[u'f^k]\,D^{n+1-k}[vf^{n+1-k}].\end{aligned}\tag{4.1}$$

On the other hand, we wish to obtain

$$D^{n+1}[uvf^{n+1}]=\sum_{k=0}^{n+1}\binom{n+1}{k}D^{k-1}[u'f^k]D^{n+1-k}[vf^{n+1-k}].$$

In this, recall that $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, whence we should have to obtain

$$\begin{aligned}D^{n+1}[uvf^{n+1}]&=\sum_{k=0}^{n}\binom{n}{k}D^{k-1}[u'f^k]\,D^{n+1-k}[vf^{n+1-k}]+\\&+\sum_{k=1}^{n+1}\binom{n}{k-1}D^{k-1}[u'f^k]D^{n+1-k}[vf^{n+1-k}].\end{aligned}\tag{4.2}$$

Comparing (4.1) and (4.2) we see that we are finished if we can just show that

$$\sum_{k=0}^{n}\binom{n}{k}D^k[u'f^k]\,D^{n-k}[vf^{n+1-k}]=\sum_{k=0}^{n}\binom{n}{k}D^k[u'f^{k+1}]\,D^{n-k}[vf^{n-k}],$$

but this is obvious at once from (1.4) if we just substitute u' for u, $v'f$ for v, and set $F=f$. Thus the proof of (1.3) is complete, since the formula is obvious for $n=0$ and $n=1$ for example.

5. Miscellaneous Remarks

It is possible to manipulate (1.3) and show that the formula may also be proved if we know that the following variation of a special case of (1.4) is true. Let

$$f_n(a,b)=\sum_{k=0}^{n}\binom{n}{k}D^k[f^{k+a}]\,D^{n-k}[f^{n-k+b}]\ .$$

Then $f_n(a+p, b) = f_n(a, b+p)$. Of course it is also quite trivial to observe that $f_n(a, b) = f_n(b, a)$.

An inductive proof of (1.2) involves a more complicated argument to show how $D^n[f^{n+1}D(uv)]$ may be found from $D^{n-1}[f^n D(uv)]$ and we leave this as an exercise for the reader.

A third formula, like (2.1) and (2.4), may be found by a further differentiation. The results is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} D^{n+1}\{[f(x)]^n u\} = \frac{utf''(z) + [1 - tf'(z)]\, u'}{[1 - tf'(z)]^3}, \tag{5.1}$$

which may be used to obtain other more complicated differentiation identities.

Examination of the proof of (1.4) in § **3** above shows that we did not need to know the complete formula for the sum of the series (2.4) but only that $u(z)$ was a factor, multiplied by something independent of $u(z)$.

Finally we offer a conjecture which we have not been able to prove. Because of the parallel between the differentiation formulas and the binomial identities, it is conjectured that an inductive proof may be given for the non-symmetrical case of (2.3) based solely upon formula (1.5). Such a proof would be the analogue of the proof in § **4** above. Inductive proofs have been given for various forms of (2.3), but none of them seem to involve (1.5). It is believed that a formula analogous to (2.3) exists which would contain both (1.2) and (1.3), but we have not yet found such a relation.

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DOUĂ EXTINDERI REMARCABILE ALE FORMULEI DE DERIVARE A LUI LEIBNIZ

(Rezumat)

Formula de derivare a lui Leibniz (1.1) poate fi extinsă în cel puţin două moduri, prin introducere unei a treia funcţii. Se consideră aici cele două cazuri (1.2), datorite lui Cauchy, şi (1.3), datorit lui T. T. Walker (în ambele $D^{-1}(Du) = u$). Ele se demonstrează cu ajutorul seriilor Taylor generalizate ale lui Lagrange.

Formula lui Walker se demonstrează de asemenea utilizînd un procedeu special de inducţie bazat pe o altă variantă a formulei generalizate a lui Leibniz. În cele din urmă se arată că pentru fiecare formulă de derivre de acest tip există o identitate binomială analogă, astfel încît cele stabilite aici dau informaţii asupra naturii unor identităţi din analiza combinatorie. În baza acestei analogii se enunţă o serie de conjecturi.

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INTÉGRATION SUR UN CONTOUR COMPORTANT UNE SINGULARITÉ DE LAURENT [1])

BY

F. BUCKENS

On se propose d'étudier l'intégrale

$$\int_{(C)} f(z)\,\mathrm{d}z, \tag{1}$$

lorsque le contour d'intégration (C) est une courbe de Jordan rectifiable, frontière d'un domaine d'analyticité (D) de $f(z)$ $\{\partial D = (C)\}$, ce contour (C) passant par un point z_0 qui est un pôle ou un point singulier essentiel de $f(z)$. On sait que lorsqu'il s'agit d'un pôle *simple* en un point à tangente *unique* sur (C) il suffit d'introduire dans la formule des résidus la moitié du résidu en ce point. Il est intéressant de considérer le cas plus général.

On peut définir cette intégrale sur le contour passant par z_0 comme la limite commune obtenue à partir de deux suites d'intégrales effectuées en contournant z_0, soit en excluant, soit en incluant ce point de l'intérieur du contour ainsi modifié, le long d'arc de cercles de rayons r de plus en plus petits.

Comme z_0 est une singularité isolée, il existe un voisinage

$$0 < |z - z_0| < R\,, \tag{2}$$

où $f(z)$ est représentée par la série de Laurent absolument convergente :

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n\,. \tag{3}$$

---

[1]) Les résultats de ce travail ont été utilisés lors de la conférence donnée le 17 avril 1971 au Département de Mathématiques de l'Institut Polytechnique de Jassy.

Nous supposons qu'un cercle de rayon $r < R$ centré en z_0 ne rencontre le contour C qu'en deux seuls points d'intersection A et B, si r est suffisamment petit. Soit α et β les arguments des sécantes $(z_A - z_0)$ et $(z_B - z_0)$ et α_0, β_0 ceux des tangentes „à gauche" et „à droite" en z_0; nous supposons que le contour C présente au voisinage de z_0 une régularité telle que (fig. 1)

$$\lim_{r=0} \alpha = \alpha_0 , \quad \lim_{r=0} \beta = \beta_0 . \tag{4}$$

Considérons la suite d'intégrales „intérieures" effectuées le long de $\overrightarrow{A c_i B}$, et correspondant à une suite de valeurs de plus en plus petites de r; de même, considérons la suite d'intégrales „extérieures" effectuées le long de $\overrightarrow{A c_e B}$. Les termes homologues (de même rayon r pour c_i et c_e) de ces deux suites ne diffèrent que de $2\pi j a_{-1}$, puisque l'intégrale extérieure peut chaque fois être considérée comme s'effectuant le long de $\overrightarrow{A c_i B} \overrightarrow{c_i A} \overrightarrow{c_e B}$; dès lors ces deux suites convergent ou divergent ensemble, et de plus, si elles convergent, elles donneront la même valeur à l'intégrale (1), puisque la différence $2\pi j a_{-1}$ est précisément le terme supplémentaire qui, d'après le théorème des résidus, apparaîtrait au second membre pour la suite des intégrales „extérieures".

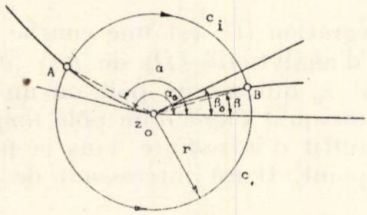

Fig. 1. — Singularité en un point anguleux de C.

Il reste à examiner dans quelles conditions ces suites convergent et d'en définir les limites éventuelles. De ce qui précède, il suffit de considérer l'une de ces suites, par exemple celle des intégrales „intérieures".

On peut, lorsque chacune de ces intégrales converge uniformément pour $r > 0$, échanger les opérateurs, intégrale et série (à convergence uniforme), lorsqu'on substitue l'expression (3) de $f(z)$ dans les intégrales. On a donc,

$$I_i = \int_{c_i} f(z)\, dz = \sum_{n=-\infty}^{\infty} a_n \int_{c_i} (z - z_0)^n\, dz, \tag{5}$$

et comme $z - z_0 = re^{j\theta}$ sur c_i:

$$
(6) \qquad \begin{aligned} I_i &= \mathrm{j} \sum_{m=-\infty}^{\infty} a_m r^{m+1} \int_{\alpha}^{\beta} \mathrm{e}^{\mathrm{j}(m+1)\theta} \, \mathrm{d}\theta = \\ &= \mathrm{j} a_{-1} (\beta - \alpha) + \sum_{m=0}^{\infty} \frac{a_m r^{m+1}}{m+1} F_{1+m}(r) - \sum_{m=2}^{\infty} \frac{a_{-m}}{(m-1) r^{m-1}} F_{1-m}(r), \end{aligned}
$$

où l'on a posé

$$
(7) \qquad F_m(r) = \mathrm{e}^{\mathrm{j}m\beta} - \mathrm{e}^{\mathrm{j}m\alpha} .
$$

Dans ces expressions α et β dépendent de r et lorqu'on fait tendre r vers zéro, convergent vers α_0 et β_0, de sorte que la première série au second membre de (6) s'évanouit. Quant à la seconde, elle diverge si

$$
a_{-m} F_{1-m}(0) \neq 0,
$$

pour l'un au moins des indices $-m = -2, -3, \ldots$

Si la singularité z_0 est un pôle simple ($a_{-1} \neq 0$, $a_{-m} = 0$ pour $m > 1$) on voit que la limite de (6) se réduit à :

$$
(8) \qquad \lim_{r=0} I_i = \mathrm{j} a_{-1} (\beta_0 - \alpha_0).
$$

Si z_0 est un pôle d'ordre $n > 1$ ($a_{-n} \neq 0$, $a_{-m} = 0$ pour $m > n$), la convergence n'est possible que si tout d'abord

$$
(9) \qquad a_{-m} F_{1-m}(0) = 0,
$$

pour tout m tel que $2 \leqslant m \leqslant n$, et cette condition implique qu'il existe des entiers k_m, au plus égaux à $m - 1$, tels que

$$
(10) \qquad a_{-m} [(m-1)(\alpha_0 - \beta_0) - 2k_m \pi] = 0,
$$

($2 \leqslant m \leqslant n$), ce qui demande qu'au moins a_{-m} ou le crochet qui le multiplie s'annulle. Mais cette condition n'est pas suffisante ; la présence de r^{m-1} aux dénominateurs de la deuxième série de (6) appelle l'application répétée de la règle de l'Hospital, et en définissant b_{pm} par :

$$
(11) \qquad b_{pm} = \lim_{r=0} \frac{\mathrm{d}^p}{\mathrm{d}r^p} F_{1-m}(r),
$$

on voit que la convergence ne peut exister que si

$$
(12) \qquad a_{-m} b_{pm} = 0, \quad (p = 1, 2 \ldots, m-2 ; \quad m = 2, 3 \ldots, n).
$$

Notons que l'application répétée de la règle de l'Hospital est légitime si on suppose que l'allure de (C) dans un voisinage de z_0 est telle que α et β et donc F_{1-m} possèdent des dérivées par rapport à r continues jusqu'à un ordre suffisant.

Compte tenu de (9), (12) se réduit à :

$$b_{1,m} = \mathrm{j}(1-m)\, e^{\mathrm{j}(1-m)\alpha_0}(\beta_0^{(1)} - \alpha_0^{(1)}) = 0,$$

$$b_{2,m} = \mathrm{j}(1-m)\, e^{\mathrm{j}(1-m)\alpha_0}[\beta_0^{(2)} - \alpha_0^{(2)} + \mathrm{j}(1-m)(\beta_0^{(1)^2} - \alpha_0^{(1)^2})] = 0,$$

etc ..., les exposants entre parenthèses indiquant les dérivations par rapport à r, et

$$\beta_0^{(1)} = \lim_{r=0} \beta^{(1)} = \lim_{r=e} \frac{d\beta}{dr} \quad \text{etc ...}$$

Dès lors, les conditions (12) se ramènent tout simplement à :

$$\alpha_0^{(p)} - \beta_0^{(p)} = 0, \quad (p = 1,\ 2 \ldots, n-2), \tag{13}$$

et comme (13) est indépendant de m, on voit que la convergence implique la disparition de tous les termes de la seconde série dans (6), sauf le dernier, pour lequel :

$$b_{n-1,n} = \mathrm{j}(1-n)\, e^{\mathrm{j}(1-n)\alpha_0}[\alpha_0^{(n-1)} - \beta_0^{(n-1)}]. \tag{14}$$

Dès lors, la limite de (6) se réduit à :

$$\lim_{r=0} I_i = -\mathrm{j}a_{-1}(\alpha_0 - \beta_0) - \frac{a_{-n}\, b_{(n-1)n}}{(n-1)!\,(n-1)}. \tag{15}$$

On remarque que la condition (13) impose à (C) des courbures égales de part et d'autre de z_0, jusqu'à l'ordre $n-2$.

Les conditions de convergence de I_i sont donc (13) et (10); cette dernière, où l'on peut poser :

$$\alpha_0 - \beta_0 = 2\lambda\pi, \quad (0 \leqslant \lambda \leqslant 1), \tag{16}$$

prend la forme

$$a_{-m}[(m-1)\lambda - k_m] = 0, \tag{17}$$

de sorte que si $a_{-m} \neq 0$, on doit avoir

$$m = 1 + \frac{k_m}{\lambda}, \tag{18}$$

m et k_m étant entiers ; ceci n'est possible que si λ est un nombre rationnel, soit q/s, sous sa forme réduite, et si k_n est un multiple de l'entier q; par exemple $k_n = h_q$, où h est entier. Dans ces conditions, (6) ne peut converger que si seuls les coefficients a_{-m} diffèrent de zéro pour lesquels, d'après (18),

$$m = (1),\ 1+s,\ 1+2s, \ldots, 1+(h-1)s, \quad n = 1 + hs. \tag{19}$$

C'est ainsi qu'en l'absence d'un point anguleux en z_0 on a $\lambda = 1/2$ de sorte que dans ces conditions (6) ne peut converger que si les coefficients a_{-m}

à indices impairs diffèrent de zéro : $a_2 = a_4 = \ldots = a_{n-1} = 0$; s'il en est ainsi la convergence existe quand la condition (13) est remplie. On en déduit que lorsque z_0 est un pôle d'ordre pair, l'intégrale I_i diverge pour $r \to 0$, lorsque la tangente en z_0 à (C) est unique.

Supposons que z_0 soit un point anguleux de (C) ; on tire de (17), pour $m = n$,

$$\lambda = \frac{q}{s} = \frac{k_n}{n-1} = \frac{hq}{n-1}, \tag{20}$$

d'où

$$s = \frac{n-1}{h}.$$

Ceci montre que si z_0 est un pôle d'ordre n, l'expression (6) ne converge que si :

1° L'angle $\alpha_0 - \beta_0$ est un multiple de $2\pi/s$, où s est lui-même un sous-multiple de $n - 1$: $\alpha_0 - \beta_0 = 2\pi/s$, $4\pi/s$,..., $2\pi(s-1)/s$;

2° La courbure de (C) de part et d'autre de z_0 présente la symmétrie exprimée par la condition (13) ;

3° La série de Laurent qui représente $f(z)$ au voisinage de z_0 ne comporte que les coefficients a_{-m} pour lesquels m prend l'une des valeurs indiquées par (19).

Les deux premières conditions sont géométriques et portent sur l'allure de (C) au voisinage de z_0, la troisième concerne le comportement de $f(z)$ au voisinage de z_0 ; si elles sont remplies, la limite de (6) est donnée par (15).

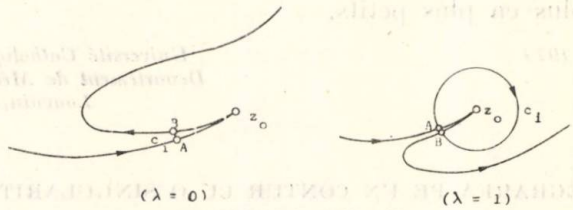

Fig. 2. — Singularité en un point de rebroussement de C.

Dans le cas où z_0 est un point de rebroussement, on peut avoir $\lambda = 0$ ou $\lambda = 1$ dans (16), (fig. 2). Dans les deux cas, on voit que la condition (13) suffit à assurer la convergence de (1) si z_0 est un pôle d'ordre n, la limite étant donnée par (15). La courbe (C) y est alors tangente à elle-même avec la même courbure jusqu'à l'ordre $n - 2$.

Enfin, si z_0 est une singularité essentielle, la convergence de I_i n'est plus possible que si, pour tout $a_{-n} \neq 0$,

$$e^{-j(n-1)\beta} - e^{-j(n-1)\alpha} = 0, \tag{21}$$

dans un voisinage de z_0, ce qui implique qu'il existe un $R > 0$ tel que pour tout r dans l'intervalle $0 \leqslant r \leqslant R$, on ait :

$$(n-1)(\alpha - \beta) = (n-1)(\alpha_0 - \beta_0) = 2k\pi, \tag{22}$$

où k doit être entier. On note que (22) se ramène à (13) où $n \to \infty$. Dans ce cas, la limite de I_i est simplement :

$$-j\, a_{-1}(\alpha_0 - \beta_0) = -\pi j 2\lambda a_{-1}.$$

On retrouve le cas d'une divergence lorsque λ dans (16) est irrationnel. Lorsque $\lambda = 1/2$ (courbe (C) à tangente continue en z_0), on doit avoir pour la convergence dans le voisinage $0 \leqslant r < R$:

$$\alpha - \beta = \alpha_a - \beta_0 = \pi,$$

tandis que d'après (19), où $s = 2$, seuls les coefficients a_{-m} à indices m impairs peuvent différer de zéro. Il en serait ainsi pour :

$$f(z) = \sinh\left(\frac{1}{z - z_0}\right) = \frac{1}{z - z_0} + \frac{1}{3!(z - z_0)^3} + \frac{1}{5!(z - z_0)^5} + \ldots$$

et dans ce cas :

$$I_i = -\pi j.$$

Il va sans dire que les conclusions précédentes résultent de la définition adoptée pour l'intégrale à travers z_0, celle de la limite commune qu'on obtient en contournant à gauche ou à droite par des arcs de cercle de rayons de plus en plus petits.

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Université Catholique de Louvain,
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Louvain, Belgique

INTEGRAREA PE UN CONTUR CU O SINGULARITATE DE TIP LAURENT

(Rezumat)

Se studiază integrala (1) cînd conturul de integrare (C) este o curbă Jordan rectifiabilă, frontiera unui domeniu de analiticitate (D) al lui $f(z)$, frontieră ce trece printr-un punct singular esenţial z_0 al funcţiei $f(z)$. Integrala prin z_0 se defineşte ca limită comună a două şiruri de integrale efectuate înconjurînd punctul z_0 (fie excluzîndu-l, ie incluzîndu-l în interiorul conturului astfel modificat) în lungul unor arce de cerc din e în ce mai mici.

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C. D. 513.83

À PROPOS D'UN THÉOREME DE POINT FIXE POUR DES APPLICATIONS MULTIVOQUES DANS LES ESPACES VECTORIELS TOPOLOGIQUES

PAR

AL. SABADÎŞ

Soit E un espace vectoriel topologique local convexe séparé. On dénote par $\delta(k) = \{x \mid x \in K, \exists$ une surface à dimensions finies F, telle que $x \in \mathrm{Fr}\,(k \cap F)\}$.

On rappelle le résultat suivant :

Théorème 1. *Soit E un espace vectoriel topologique local convexe séparé, K un de ses sous-ensembles, non-vide, convexe et compact, et T une application multivoque de K en E, supérieurement semi-continue et avec $Tx \neq \neq \emptyset$, fermé et convexe pour $\forall x \in K$. On suppose que pour tout $u \in \delta(K)$ il y a un élément $W \in T(u)$, un élément $u_1 \in K$ et un nombre réel $\lambda > 0$ tel que $w - u = \lambda(u_1 - u)$. Il existe allors un élément u_0 dans K tel que $u_0 \in T(u_0)$.*

À l'aide de ce théorème on démontre le

Théorème 2. *Soit E un espace vectoriel topologique local convexe, K un de ses sous-ensembles non-vide, convexe et compact, et T une application multivoque de K dans E, semi-continue supérieurement et avec $Tx \neq \emptyset$, fermé et convexe pour tout $x \in K$. On suppose que ponr tout $x \in \delta(K)$, $Tx \cap \cap K \neq \emptyset$. Il existe alors un élément $u_0 \in K$ tel que $u_0 \in T(u_0)$.*

Démonstration. Pour tout $u \in \delta(k)$, il y a $w \in T(u) \cap K$, $u_1 = w \in K$ et $\lambda = 1$ pour lesquels $w - u = \lambda(u_1 - u) = w - u$. Cela indique qu'on peut appliquer le Théorème 1. Il y aura donc un élément $u_0 \in K$ tel que $u_0 \in T(u_0)$ et la démonstration s'achève ainsi.

Corollaire 1 (théorème de Ky Fan). *Soit E un espace vectoriel topologique local convexe et K un sous-ensemble convexe et compact de E. Si T est une application multivoque de K dans K supérieurement semi-continue*

et telle que pour tout $x \in K$ $T(x)$ est non-vide et convexe, il existe alors un point $x_0 \in K$ tel que $x_0 \in T(x_0)$.

Démonstration. On observe que dans ce cas $T(x) \cap K = T(x) \neq \emptyset$ pour $\forall x \in \delta(K) \subset K$. Il en résulte qu'on peut recourir au Théorème 2 et il y aura un élément $x_0 \in K$ tel que $x_0 \in T(x_0)$.

Corollaire 2 (théorème de Kakutani, forme forte). *On considère l'espace R^n et K un de ses sous-ensembles convexe et compact. Soit T une application multivoque supérieurement semi-continue de K en lui-même avec $T(x) \neq \emptyset$ et convexe, pour tout $x \in K$. Il existe alors un point $x_0 \in K$, pour lequel $x_0 \in Tx_0$.*

La démonstration est immédiate si l'on tient compte de ce que R^n est un espace local convexe.

Corollaire 3 (théorème de Kakutani, forme faible). *Soit $[a, b] \subset R$ et I une application multivoque supérieurement semi-continue de $[a, b]$ en lui-même telle que $T(x)$ est un sous-intervalle fermé de $[a, b]$, pour tout $x \in [a, b]$. Il y a alors un point $x_0 \in [a, b]$ tel que $x_0 \in T(x_0)$.*

La demonstration résulte du fait qu'un intervalle fermé de R est local convexe, par l'application du Corollaire 2.

Corollaire 4 (théorème de Shauder-Tikhonov). *Si E est un espace topologique local convexe et $K \subset E$ est convexe et compact et f une application univoque continue de K en lui-même, il existe alors un point $x_0 \in K$ tel que $f(x_0) = x_0$.*

Démonstration. On recourt au Corollaire 1 et on rappelle qu'une application univoque continue peut être considérée comme une application supérieurement semi-continue puisque pour tout ensemble ouvert $G \subset K$ avec $f(x) \in G$ il existe un voisinage $V(x) \subset K$ tel que $\forall x \in V(x_0), f(x) \subset G$.

Corollaire 5 (théorème de Brouwer, forme forte). *Soit K un ensemble non-vide convexe et compact de R^n et f une application univoque continue de K dans K. Il existe alors un point $x_0 \in K$ tel que $x_0 = f(x_0)$.*

La démonstration résulte du Corollaire 2, en interprétant f comme une application supérieurement semi-continue.

Corollaire 6 (théorème de Brouwer, forme faible). *Si f est une application univoque continue de l'intervalle $[a, b]$ en lui-même, il existe alors un point $x_0 \in [a, b]$ tel que $f(x_0) = x_0$.*

En effet, si l'on dénote $f(x) = y$ on peut considérer $f(x) = [y, y]$ un intervalle fermé et l'on applique ensuite le Corollaire 3.

Ce corollaire exprime le fait connu que le graphique d'une application continue de l'intervalle $[a, b]$ en lui-même coupe la première bissectrice au moins une fois.

Corollaire 7. *Si T est une application multivoque continue de l'intervalle $[a, b]$ en lui-même telle que $x \in [a, b]$, $T(x) \neq \emptyset$ et fermé, il existe un point $x_0 \in [a, b]$ tel que $x_0 \in T(x_0)$ et $x_0 = \max T(x_0)$.*

D é m o n s t r a t i o n. On emploiera le théorème de maximum suivant. Si $\varphi(y)$ est une fonction numérique continue dans Y et si T est une application continue de X tel que $T(x) \neq \emptyset$ pour tout x, alors $M(x) = \max \{\varphi(y) \mid y \in T(x)\}$ est une fonction numérique continue dans X. Si l'on pose $\varphi = I$, il en résulte que $M(x) = \max T(x)$ est une application univoque continue dans X.

On peut maintenant démontrer le Corollaire. L'application $M(x) = \max T(x)$ est continue sur $[a, b]$. En vertu du corollaire 7 il y aura un point $x_0 = \max M(x_0) \in T(x_0)$, $T(x_0)$ étant compact.

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Institut Polytechnique, Jassy,
Chaire de Mathématiques I

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ASUPRA UNEI TEOREME DE PUNCT FIX PENTRU APLICAŢII MULTIVOCE ÎN SPAŢII VECTORIALE TOPOLOGICE

(Rezumat)

Se demonstrează unele teoreme clasice de punct fix, utilizînd în acest scop teoremele 1 şi 2.

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G. D. 517.2

SUR UN CAS DE VALIDITÉ DE LA FORMULE DE GREEN

PAR

ANTOINE PACQUEMENT

1. Introduction

On sait que pour qu'une forme de classe C_1, $\omega = u\,\mathrm{d}x + v\,\mathrm{d}y$ dans R^2 soit fermée, il faut et il suffit que l'on ait l'égalité

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{1.1}$$

Nous ne connaissons pas d'énoncé dans le cas où $\partial v/\partial x$ et $\partial u/\partial y$ sont seuls supposés exister.

Pour prouver la suffisance de (1.1) dans ce cas, nous supposerons l'existence des dérivées partielles $\partial v/\partial x$ et $\partial u/\partial y$. Nous construirons deux suites de fonctions (u_n) et (v_n) tendant respectivement vers u et v dans un rectangle K_0, telles que:

a) les dérivées partielles $\partial v_n/\partial x$ et $\partial u_n/\partial y$ soient sommables dans K_0 et égales ;

b) l'on puisse passer à la limite dans la formule

$$\int_{\Delta K_0} u_n\,\mathrm{d}x + v_n\,\mathrm{d}y = 0.$$

Pour cela, on définira les suites (u_n) et (v_n) de façon à

c) être uniformément bornées sur ΔK_0 ;

d) avoir des dérivées $\partial v_n/\partial x$, $\partial u_n/\partial y$ tendant vers $\partial v/\partial x$ et $\partial u/\partial y$.

Les sections v_y (respectivement u_x) seront des totales indéfinies continues de x (respectivement de y), donc A. C. G.

Pour étudier ces fonctions nous introduirons la notion de *série-A*, séries ne comportant pour toutes valeurs des variables qu'un nombre fini de termes non nuls.

Avant prouvé la suffisance de (1.1) on passera à un cas où la divergence est non nulle.

2. Rappels concernat les fonctions A. C. et A. C. G.

2.1. Notations

On utilisera pour une fonction d'intervalle rectangulaire $K \to F(K) \in \in R$ les notations

$$K = [x_1, x_2 ; y_1, y_2], \quad K \subset K_0 = [a_1, a_2 ; b_1, b_2],$$

ainsi que la notation $F(M)$ $(M \in K_0)$ pour la fonction de point correspondante.

On passe aisément de la première notion à la seconde en posant

$$F(M) = \Delta[F ; a_1, x ; b_1, y],$$

$F(K) = \Delta[F ; x_1, x_2 ; y_1, y_2] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1 y_1)$. Les sections de F par $x = x_0$ (respectivement $y = y_0$) seront les fonctions de point

$$F_{x_0} : y \to F(x_0, y), \quad F_{y_0} : x \to F(x, y_0).$$

2.2. Définitions

a) *Normes.* Les normes seront celles de la convergence uniforme.

b) *Fonctions A. C. sur un ensemble.* F sera dite A. C. sur l'ensemble E, s'il existe une fonction Φ, A. C. dans K_0, telle que $F \mid E = \Phi \mid E$.

c) *Fonctions A. C. G.* F sera dite A. C. G. si elle est continue et s'il existe une famille dénombrable de fermés (E_n) tels que, Φ_0 étant A. C. dans K_0, on ait

$$\bigcup_n E_n = K_0, \quad F \mid E_n = \Phi_n \mid E_n, \quad \forall n \in N.$$

d) *Fonction A. C* (ou A. C. au sens restreint) sur* $E \subset K_0$. E étant fermé, (U_n) désignant les composantes connexes de $K_0 - E$, $\Omega(F, U_n)$ l'oscillation de F dans U_n, F sera dite A. C* sur E si :

i) F est A. C. sur E ; ii) $\sum_n \Omega(F, U_n) < +\infty$.

2.3. Propriétés des fonctions A. C. et A. C. G.

Si $F(K)$ est A. C., il existe une fonction L-intégrable f telle que

$$\forall K \subset K_0, \; F(K) = \iint_K f \, dx \, dy = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} f_x dy = \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f_y \, dx,$$

en vertu des théorèmes de Fubini et de Radon-Nikodym.

On déduit de cette remarque et des définitions la

Proposition 1. *Si F est A. C. G. dans K_0 :*

1) *Les sections F_x, F_y sont A. C. G. ;*

2) *F possède p. p. une dérivée partielle approximative en x (respectivement en y) et des dérivées secondes approximatives mixtes égales à f.*

Ces dérivées sont égales aux dérivées ordinaires là où celles-ci existent.

Nous aurons besoin de la propriété suivante

Proposition 2. *Si la fonction $\varphi(M)$ est A.C., il en est de même de*

$$\Phi(M) = \int_{x_1}^{x} \varphi(s, y)\, ds.$$

La démonstration se fait aisément en recourant aux suites monotones.

3. Series-A et A-suites

Définition 1. La série

$$s = \sum_{1}^{\infty} w_n, \quad w_n : K_0 \to R$$

sera dite *série-A*, si les deux conditions suivantes sont remplies:

i) $\forall M \in K_0,\ \exists n_0(M) : n > n_0 \to w_n(M) = 0$;

ii) si $A_n = \{M \in K_0,\ w_{n+p}(M) = 0,\ \forall p > 0\}$, il existe un fermé B_n tel que

$$\forall n,\ B_n \subset A_n,\ \bigcup_n B_n = K_0;$$

on écrira : $s = \sum_A w_n$.

Si les (w_n) sont continues et vérifient (i) elles vérifient (ii).

Aux séries-A correspondent des A-suites (u_n). La condition (i) s'écrit alors

$$\forall M \in K_0, \quad \exists n_0(M) : n > n_0, \quad u_n(M) = u_{n_0}(M).$$

Si $u_n \to u$ quand $n \to \infty$ on écrira $u = \lim_A u_n$.

On a la propriété suivante:

Proposition 1. *S'il existe une suite de fermés (E_n) et de fonctions continues (F_n), tels que*

$$\bigcup_n (E_n) = K_0, \quad F \mid E_n = F_n \mid E_n,$$

les F_n possédant dans K_0 la propriété: toutes les sections $F_{n\xi}$ (respectivement $F_{n,\eta}$) sont A. C. dans $[b_1, b_2]$ (respectivement $[a_1, a_2]$), alors F peut être représentée comme limite d'une A-suite (u_n) dont les termes ont les mêmes propriétés que les (F_n).

Remarque 1. Si F est bornée, pour toute A-suite telle que $F = \lim\limits_{A} u_n$, on peut prendre $\|u_n\| < \|F\|$.

On a l'importante propriété ci-après :

P r o p o s i t i o n 2. *Une série-A de fonctions $A.C.$ peut être dérivée approximativement terme à terme presque partout.*

C o r o l l a i r e 1. *Soit $s = \sum\limits_{a} u_n$ une série-A de fonctions définies dans un intervalle I_0 de R, telle que s et chaque fonction u_n soient dérivables, alors on peut dériver la série s terme a terme.*

Par contre il n'y a aucune raison pour que les primitives des termes d'une série-A (respectivement d'une A-suite) forment une série-A (respectivement une A-suite).

On a cependant l'énoncé suivant dans R^2 :

P r o p o s i t i o n 3. *Si $s = \sum\limits_{A} u_n$ provient de la dérivation terme à terme (par rapport a y) de la série-A*

$$S = \sum_{A} U_n,$$

ou S et les U_n sont continues, S_x $A.C.G.$, U_n égale a une fonction dont les sections $U_{n,x}$ sont $A.C.$, alors on peut intégrer la série s terme à terme par rapport à y.

Il est utile de savoir dans quels cas une fonction f peut être développée en série-A, justiciable d'une intégration terme à terme.

P r o p o s i t i o n 4. *Si f, définie dans K_0, satisfait aux conditions suivantes :*

a) *Sur tout ensemble fermé $H \subset K_0$, il existe une portion*

$$\pi = H \cap K$$

telle que $f \mid \pi_x$ soit L-intégrable en y pour tout x ;

b) *f_x possède une primitive en y pour presque tout x, F_x, qui soit la section d'une fonction F continue dans K_0, et telle que dans chaque portion π définie en* a), *F soit égale a une fonction continue $\widetilde{F}_\pi$, pour laquelle les sectttns $(\widetilde{E}_\pi|\pi)_x$ sont $A.C.$ en y.*

Alors f peut être représentée comme une série-A intégrable terme à terme en y.

Il résulte en effet du théorème de B a i r e que F vérifie la Proposition 2.

Le développement de f en série-A, intégrable terme à terme est lié à l'obtention d'une partition dénombrable de K_0 en partitions (π_r) disjointes, sur chacune de celles-ci f est la dérivée d'une fonction continue F_n dont les sections $F_{n,x}$ sont A. C. en y.

4. Condition de validité de la formule de Green

Nous supposerons d'abord la divergence nulle et chercherons une partition dénombrable $K_0 = \bigcup_r \pi_r$ valable pour les deux fonctions $\partial v/\partial x$ et $\partial u/\partial y$, ce qui permettra d'exprimer ces fonctions comme limites de A-suite, pour lesquelles le passage à la limite sur $\int$ soit licite.

L'obtention des (π_r) repose sur les propriétés suivantes :

P r o p o s i t i o n 1. *Sur tout fermé H, il existe une portion $\pi = H \cap K$ où chaque dérivée partielle (finie) d'une fonction continue est bornée.*

Nous aurons aussi à utiliser la Proposition suivante qui est démontrée dans le traité de S a k s.

P r o p o s i t i o n 2. *Si la fonction continue u est dérivable en y, sur tout fermé H, il existe une portion π, ou u_x est $A.C^*$ en y, et cela uniformément par rapport à x.*

Ces propriétés entraînent la conséquence suivante :

C o r o l l a i r e 1. *Si u continue possède partout une dérivée partielle finie $\partial u/\partial y$, alors cette dérivée vérifie les conditions de la Proposition 4 du* § **3**

C'est là un point assez délicat à démontrer ; comme il exige d'assez longs développements, nous admettrons ce corollaire.

On peut alors entreprendre la démonstration du

T h é o r è m e 1. *Si les fonctions u et v sont continues dans K_0, et possèdent des dérivées partielles finies $\partial u/\partial y$ et $\partial v/\partial x$ égales, alors la forme*

$$\omega = u\,\mathrm{d}x + v\,\mathrm{d}y \tag{4.1}$$

est fermée dans K_0.

La d é m o n s t r a t i o n sera divisée en plusieurs parties.

a) *Détermination des portions (π_r), des suites (α_n) et (β_n).* D'après la Proposition 1, sur tout fermé $H \subset K_0$, il existe une portion

$$\pi = H \cap K \quad (K \in K_{i(i \in I)} \text{ base de la topologie de } K_0),$$

sur laquelle $\partial u/\partial y$, $\partial v/\partial x$ sont bornées, et $u|_\pi$ et $v|_\pi$ simultanément A. C., ainsi que les sections $\left.\dfrac{\partial u}{\partial y}\right|_{\pi_x}$, $\left.\dfrac{\partial y}{\partial x}\right|_{\pi_y}$.

Il s'ensuit que u peut être représentée comme limite à une A-suite (α_n), telle que

$$u = \lim_A \alpha_n, \qquad \|\alpha_n\| < \|u\| \tag{4.2}$$

et par conséquent

$$\frac{\partial u}{\partial y} = \lim \frac{\partial \alpha_n}{\partial y}. \tag{4.3}$$

D'après la démonstration de la Proposition 4 du § **3**, l'on peut trouver une famille au plus dénombrable de portions π, $(\pi_r)_{r \in N}$, disjointes, telles que

$$K_0 = \bigcup_r \pi_r$$

et que pour tout r on puisse trouver un indice n_r tel que

$$n > n_r \to \alpha_n |_{\pi_r} = \alpha_{n_r} |_{\pi_r}.$$

Nous déterminerons alors la suite (β_n) par l'égalité dans π_r

$$\frac{\partial \beta_n}{\partial x} = \frac{\partial \alpha_n}{\partial y}. \tag{4.4}$$

Les $\partial \beta_n / \partial x$ forment une A-suite, et restent comme les (α_n) égaux dès que $n \geqslant n_r$ à $\partial \beta_{n_r} / \partial x$.

Montrons que la suite $(\partial \beta_n / \partial x)$ vérifie les conditions de la proposition 4 du § **3**.

La définition des (π_r) montre que la condition a) de cette proposition est satisfaite.

La A-limite des $\partial \beta_n / \partial x$ est d'après (4) la même que celle des $\partial \alpha_n / \partial y$, c'est-à-dire $\partial u / \partial y = \partial v / \partial x$. On a donc

$$\lim_A \frac{\partial \beta_n}{\partial x} = \frac{\partial v}{\partial x};$$

v d'autre part est continue dans K_0, par hypothèse ; et elle est la primitive de $\partial v / \partial x$ pour toutes les valeurs de y.

Par définition des π_r, $v |_\pi$ est la restriction à π d'une des fonctions $v |_{\pi_r}$, qui sont A. C. La condition b) de la Proposition 4 du § **2** est ainsi vérifiée.

Ainsi la A-suite $(\partial \beta_n / \partial x)$ est justiciable du passage à la limite sous $\int$ et la suite (β_n) une A-suite de limite v. On aura donc d'après (4.4)

$$\int_{\Delta K_0} \alpha_n \mathrm{d}x + \beta_n \mathrm{d}y = 0. \tag{4.5}$$

b) *Détermination des suites* (u_n) *et* (v_n). i) Les fonctions v_x sont A.C.G., et en particulier v_{a_2}.

D'après le § **3**, corollaire 1, dans R, on peut représenter v_{a_2} comme limite d'une A-suite (w_n) de fonctions de y A. C. uniformément bornées, soit

$$v_{a_2} = \lim_A w_n, \quad \| w_n \| < \| v_{a_2} \|. \tag{4.6}$$

Choisissons $c \in [a_1, a_2]$, v_2 est A. C. G. en y.

Nous allons modifier α_n et β_n dans la bande $[c, a_2] \times [b_1, b_2]$, n étant provisoirement fixe. On introduira deux nouvelles fonctions (qui dépendent de c), $\widetilde{u}_n$ et $\widetilde{v}_n$ telles que

$$\widetilde{u}_n(a_2, y) = w_n(y), \qquad \widetilde{u}_n(c, y) = \beta_n(c, y)$$

et on prendra $\widetilde{v}_{n,y}$ linéaire et continue en x, pour $x \in [c, a_2]$ d'où

$$\frac{\partial \widetilde{v}_n}{\partial x} = \frac{w_n(y) - \beta_n(c, y)}{a_2 - c}, \tag{4.7}$$

quantité indépendante de x. Puis on déterminera u_n par la conditon

$$\frac{\partial \widetilde{u}_n}{\partial y} = \frac{\partial \widetilde{v}_n}{\partial x}, \qquad \widetilde{u}_n(x, b_1) = 0. \tag{4.8}$$

Les deux intégrales suivantes sont bien définies

$$\int_{\Delta K_0} \widetilde{v}_n \mathrm{d}y = \int_{b_1}^{b_2} \widetilde{v}_n(a_2, y)\,\mathrm{d}y, \qquad \int_{\Delta K_0} \widetilde{u}_n \mathrm{d}x = \int_{a_1}^{a_2} \widetilde{u}_n(x, b_2)\,\mathrm{d}x. \tag{4.9}$$

Il suffit de montrer que cela est vrai des intégrales doubles égales

$$\iint_{K_0} \frac{\partial \widetilde{v}_n}{\partial x}\,\mathrm{d}x\,\mathrm{d}y, \qquad \iint_{K_0} \frac{\partial \widetilde{u}_n}{\partial y}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.10}$$

Prenons la première intégrale. Si

$$x \in [a_1, c], \qquad \frac{\partial \widetilde{v}_n}{\partial x} = \frac{\partial \beta_n}{\partial x} = \frac{\partial \alpha_n}{\partial y},$$

or cette fonction est sommable en vertu de la proposition 2 du § **2**; $x \in [c, a_2]$, $\dfrac{\partial \widetilde{v}_n}{\partial x}$ est la fonction de y définie par (7), continue dans $[c, a_2] \times \times [b_1, b_2]$.

Ainsi les intégrales doubles sont celles de fonctions L-intégrables et peuvent être calculées par intégrations successives, d'ou l'existence des intégrales (4.9).

En appliquant le théorème de F u b i n i à la différence des intégrales (4.10) on trouve facilement

$$\int_{\Delta K_0} \widetilde{u}_n \mathrm{d}x + \widetilde{v}_n \mathrm{d}y = 0. \tag{4.11}$$

ii) On va pouvoir des fonctions $\widetilde{u}_n$ et $\widetilde{v}_n$ déduire les suites (u_n) et (v_n) cherchées.

Donnons à c une suite croissante de valeurs (c_p) tendant vers a_2. À chacune de ces valeurs, pour n fixé, correspondent des fonctions $\widetilde{u}_n$ et $\widetilde{v}_n$ différentes, qu'on note $\widetilde{u}_{n,p}$ et $\widetilde{v}_{n,p}$. On notera

$$u_n = \widetilde{u}_{n,n}, \qquad v_n = v_{n,n}$$

et (u_n) et (v_n) sont des A-suites, on aura

$$u = \lim_A u_n, \qquad v = \lim_A v_n, \tag{4.12}$$

et en vertu de (4.11)

$$\int_{\Delta K_0} u_n \mathrm{d}x + v_n \mathrm{d}y = 0. \tag{4.13}$$

c) *Démonstration de la formulle de G r e e n.* Il faut justifier le passage à la limite dans la formule (4.13), qui s'écrit

$$\int_{a_1}^{a_2} u_n(s, b_2)\,\mathrm{d}s = \int_{c_1}^{b_2} v_n(a_2, t)\,\mathrm{d}t. \tag{4.14}$$

Or on a par définition (4.6)

$$\int_{b_1}^{b_2} v_n(a_2, t)\,\mathrm{d}t = \int_{b_1}^{b_2} w_n(t)\,\mathrm{d}t,$$

et les (w_n) étant uniformément bornées, de A-limite v, on a

$$\lim_{n\to\infty} \int_{b_1}^{b_2} v_n(a_2, t)\,\mathrm{d}t = \int_{b_1}^{b_2} v(t)\,\mathrm{d}t. \tag{4.15}$$

Étudions la première des intégrales (5.14).

On a, dès que $x \leqslant c_n$, $u_n(x, y) = \widetilde{u}_{n,n}(x, y) = \alpha_n(x, y)$, d'où pour $c_n \geqslant \geqslant x_j \geqslant x_i$, les α_n étant uniformément bornés (4.2)

$$\left| \int_{x_i}^{x_j} u_n(s, b_2)\,\mathrm{d}s \right| = \int_{x_i}^{x_j} \alpha_n(s, b_2)\,\mathrm{d}s < \| u \| (x_j - x_i). \tag{4.16}$$

Par ailleurs pour $x \geqslant c_n$, on a

$$u_n(x,\ b_2) = \int_{b_1}^{b_2} \frac{\partial u_n}{\partial y}(x,\ y)\,\mathrm{d}y = \int_{b_1}^{b_2} \frac{\partial \widetilde{v}_{n,\,n}}{\partial x}\,\mathrm{d}y = \varphi(c_n),$$

quantité indépendante de x. On a donc pour $x \geqslant c_n$, $u_n(x,\ b_2) = u_n(c_n,\ b_2)$ et puisque u_n est égale à α_n pour $x = c_n$, (et $\| \alpha_n \| \leqslant \| u \|$)

$$\left| \int_x^{a_2} u_n(s,\ b_2)\,\mathrm{d}s \right| \leqslant \| u \| (a_2 - x). \tag{4.17}$$

De (5.16) et (5.17) on déduit pour $x_i,\ x_j \in [a_1,\ a_2]$

$$\left| \lim \int_{x_i}^{x_j} u_n(s,\ b_2)\,\mathrm{d}s \right| \leqslant \| u \| (x_j - x_i), \tag{4.18}$$

ce qui prouve la continuité de cette limite sur $[a_1,\ a_2]$.

Par ail leurs la quantité $\int_{a_1}^{x} u(s,\ b_2)\,\mathrm{d}s$ est une fonction continue de x et comme pour $x < a_2$:

$$\lim_{n\to\infty} \int_{a_1}^{x} u_n(s,\ b_2)\,\mathrm{d}s = \lim_{n\to\infty} \int_{a_1}^{x} \alpha_n(s,\ b_2)\,\mathrm{d}s = \int_{a_1}^{x} u(s,\ b_2)\,\mathrm{d}s$$

puisque les $|\alpha_n|$ sont bornées dans leur ensemble, on a par continuité des deux membres

$$\lim_{n\to\infty} \int_{a_1}^{a_2} n_n(s,\ b_2)\,\mathrm{d}s = \int_{a_1}^{a_2} u(s,\ b_2)\,\mathrm{d}s, \tag{4.19}$$

ce qui justifie le passage à la limite dans la première intégrale.

On peut donc passer à la limite dans (4.13) et la formule de Green est prouvée.

Corollaire 2. *Si u et v sont continues dans K_0, et si $\dfrac{\partial u}{\partial y}$ et $\dfrac{\partial v}{\partial x}$ sont partout finies, et telles que la divergence*

$$\delta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

soit continue, on a la formule de Green pour $K \subset K_0$

$$\iint_K \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y = \int_{\Delta K} u\,\mathrm{d}x + v\,\mathrm{d}y.$$

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ASUPRA UNUI CAZ DE VALIDITATE A FORMULEI LUI GREEN

(Rezumat)

Se ştie că egalitatea (1.1) este necesară şi suficientă pentru ca forma $\omega = u\mathrm{d}x + v\mathrm{d}y$ din clasa C_1 să fie închisă în R^2. Se demonstrează suficienţa condiţiei (1.1) pentru cazul cînd se admite numai existenţa derivatelor $\partial v/\partial x$ şi $\partial u/\partial y$, introducîndu-se în acest scop noţiunea de serii-A (studiate în § 3).

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ON THE CONVOLUTION OF DISTRIBUTIONS (I) [1]

BY

P. CRĂCIUNAŞ

0. Introduction

In a set of previous papers, presented at the scientific sessions of the Polytechnic Institute of Cluj, University and Polytechnic Institute of Jassy during the years 1969 and 1970, we have dealt with some problems related to the convolution and the Fourier transform of distributions in the sense of Mikusinski [1]. These problems as well as some considerations on the equations of convolution in the above mentioned sense, will be presented in our dissertation [9].

Two papers related with Fourier transform of distributions are already printed [7], [8].

In what follows we present some results refered to the convolution of distributions and to the equations of convolution.

1. Generalities. The Convolution of Distributions

Definition 1. A sequence $\{f_n(x)\}$ of indefinitely differentiable functions is fundamental in an open set O if for any open relatively compact set $\Omega \subset O$, there exists a multi index $k \geqslant 0$ and a sequence of indefinitely differentiable functions $\{F_n(x)\}$ so that in Ω we have $f_n(x) = F_n^{(k)}(x)$ and $\{F_n(x)\}$ is uniformly convergent.

It is not difficult to show that this definition is equivalent with the definition of a fundamental sequence given in [1], but if we will use this definition, we make it because it don't requires any restriction that the compact support of one of the distributions in $f(x) * g(x)$ be an interval $[a, b]$.

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[1]) Presented at the scientific session of the Polytechnic Institute of Cluj, september 1969.

So, as in [1], if $\alpha(x) \in C_0^\infty(R^\nu)$ and $f(x)$ is a distribution given by the fundamental sequence $\{f_n(x)\}$, then the convolution $\alpha(x) * f(x)$ is given by the fundamental sequence $\{\alpha(x) * f_n(x)\}$, which defines an indefinitely differentiable function.

The convolution between two distributions having their supports on $[0, \infty)$ is defined in [3], but some more general results refering to the convolution of distributions in the sense of Mikusinski can be found in [6]. Now we define the distributions with compact support and we prove the equivalence of these results with that of Schwartz's theory of distributions [2].

Definition 2. Let us say that the distribution $g(x) = [g_n(x)]$ has a compact support K_0 if in its equivalence class it contains at least one fundamental sequence of indefinitely differentiable functions $\{g_n(x)\}$, with compact supports and such that for any neighborhood V of K_0 we will have $K_0 \subset \operatorname{supp} g_n(x) \subset V$, the last inclusion following to be true not for any n but for all n greater then an $N_0 = N(V)$.

Theorem 1. *If $g(x)$ is a distribution with compact support K_0, then we have $g(x) = \sum_{|k| \leqslant m} G_k^{(k)}(x)$, where $G_k(x)$ are continuous functions with their supports in an arbitrary neighborhood V of k_0.*

Proof. If Ω is an arbitrary neighborhood of K_0 with $\bar{\Omega} \subset V$ then will exists an index $m \geqslant 0$ and a sequence $\{G_n(x)\}$ of continuous indefinitely differentiable functions such that in Ω we have $g_n(x) = G_n^{(m)}(x)$ and $G_n(x) \underset{u}{\rightarrow} G(x)$.

If now $\alpha(x)$ is an indefinitely differentiable function with compact support contained in Ω, and $\alpha(x) = 1$ on an arbitrary neighborhood of K_0, then $\alpha(x) G_n(x)$ will have compact supports contained in V and for $n \geqslant N(\alpha)$ we have

$$g_n(x) = \alpha(x) G_n^{(m)}(x) = \sum_{k \leqslant m} \{(-1)^{|k|} C_m^k G_n(x) \alpha^{(m-k)}(x)\}^{(k)} \tag{1}$$

and then

$$\begin{aligned} g(x) = [g_n(x)] &= [\sum_{k \leqslant m} \{(-1)^{|k|} C_m^k G_n(x) \alpha^{(m-k)}(x)\}^{(k)}] = \\ &= \sum_{k \leqslant m} \{(-1)^{|k|} C_m^k G(x) \alpha^{(m-k)}(x)\}^{(k)} = \sum_{k \leqslant m} G_k^{(k)}(x), \end{aligned} \tag{1'}$$

where $G_k(x) = (-1)^{|k|} C_m^k G(x) \alpha^{(m-k)}(x)$ are continuous functions having compact supports contained in V.

Theorem 2. *Being given the sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ such that $f_n(x) \rightrightarrows f(x)$ in R^ν and $g_n(x)$ have compact supports and $g_n(x) \underset{u}{\rightarrow} g(x)$, such that supp $g_n(x) \subset V$ for any compact neighborhood V of supp $g(x)$ and any $n \geqslant N(V)$, then $f_n(x) * g_n(x) \rightrightarrows f(x) * g(x)$ in R^ν.*

Proof. Let us be O' a relatively compact open set of R^ν and O with the same property and such that $O' - O \subset V$. Then the sequence

$\{f_n(t)g_n(x-t)\}$ is uniformly convergent to $f(t)\,g(x-t)$ in O' and by integration it results the statement of the theorem.

Definition and Theorem 3. *Being given two distributions $f(x)$ and $g(x)$, one, for* example $g(x)$, *having compact support, then if $f(x) = [f_n(x)]$ and $g(x) = [g_n(x)]$, with $\{g_n(x)\}$ satisfying to definition 2, it follows that we can define the convolution $f(x) * g(x)$ as being the distribution $h(x)$ given by the fundamental sequence $h_n(x) = f_n(x) * g_n(x)$.*

Proof of the consistence. Firstly, it is not difficult to see that the sequence $\{h_n(x)\}$ is fundamental, because if we take a relatively compact open set O' and O with the same property and such that $O' - O \subset V$, then in O $f_n(x) = F_n^{(p)}(x)$ and $F_n(x) \underset{u}{\rightarrow} F(x)$, as for $g_n(x)$ there exists an index $m \geqslant 0$ and $\{G_{n,k}(x)\}$ such that $g_n(x) = \sum_{k \leqslant m} G_{n,k}^{(k)}(x)$, $G_{n,k}(x) \underset{u}{\rightarrow} G_k(x)$ in V and $G_{n,k}(x)$ and $G_k(x)$ have their compact supports contained in V (see the proof of the theorem 1). Then finally in O', we have

$$h_n(x) = f_n(x) * g_n(x) = F_n^{(p)}(x) * \sum_{k \leqslant m} G_{n,k}^{(k)}(x) = \sum_{k \leqslant m} (F_n(x) * G_{n,k}(x))^{(p+k)}$$

and $F_n(x) * G_{n,k}(x) \underset{u}{\rightarrow} F(x) * G_k(x)$. Since the supports of $F_n(x) * G_{n,k}(x)$ are arbitrary sets, we can substitute the finite sum of derivatives with a only one derivative such that $h_n(x) = H_n^{(l)}(x)$ and $H_n(x) \underset{u}{\rightarrow}$ in O'.

Ultimately, if $\{f_n(x)\} \sim \{f'_n(x)\}$, $\{g_n(x)\} \sim \{g'_n(x)\}$ and $\{g'_n(x)\}$ satisfies also to the definition 2, then the sequences $f_1(x)$, $f'_1(x)$,..., $g_1(x)$, $g'_1(x)$, ... are also fundamental and the ast satisfies to the definition 2. But then it results that the sequence $f_1(x)$ $g_1(x)$, $f'_1(x) * g'_1(x)$,... is fundamental and hence the equivalence of the sequences $\{f_n(x) * g_n(x)\}$ and $\{f'_n(x) * g'_n(x)\}$.

Consequences. 1. The convolution is comutative and if at least two distributions have compact supports, the product $f(x)$ $g(x)$ $h(x)$ is associative.

2. $(f(x) * g(x))^{(k)} = f^{(k)}(x) * g(x) = f(x) * g^{(k)}(x)$.

3. If $f(x)$ and $g(x)$ have compact supports, $f(x) * g(x)$ has also a compact support and $\operatorname{supp} f(x) * g(x) = \operatorname{supp} f(x) + \operatorname{supp} g(x)$. In general $\operatorname{supp} f(x) * g(x) \subset \operatorname{supp} f(x) + \operatorname{supp} g(x)$.

4. If $\delta(x) = [\delta_n(x)]$ as in [1], then it results $f(x) * \delta^{(k)}(x) = f^{(k)}(x)$ for any $k \geqslant 0$.

Theorem 4. *If we denote by $\mathcal{M}'_0$ the set of the distributions defined as in definition 2, then there exists a one to one corespondence between $\mathcal{M}'_0$ and $\mathcal{E}'$* [2].

Proof. Let us be $f(x) = [f_n(x)]$ a distribution of $\mathcal{M}'_0$, $K_0 = \operatorname{supp} f(x)$, and $\alpha(x) \in C_0^\infty$ such that $\alpha(x) = 1$ in a compact neighborhood V of K_0. Then, because for $n \geqslant N(V)$ $\operatorname{supp} f_n(x) \subset V$, for any $\varphi(x) \in \mathcal{E}$ we can write

$$[f_n(x)](\varphi) = \lim_{n\to\infty} \int\!\!\int \ldots \int_{R^\nu} f_n(x)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} \int\!\!\int \ldots \int_{R^\nu} f_n(x)(\alpha(x)\,\varphi(x))\,\mathrm{d}x$$

and the last limit will exists because $\{f_n(x)\}$ is a fundamental sequence and $\alpha(x)\,\varphi(x) \in C_0^\infty(R^\nu)$.

It is easy to see that the limit don't depends of $\alpha(x)$ and of the choise of the sequence $\{f_n(x)\}$ by the equivalence class of $f(x)$. Also the limit is linear and continuous with respect to $\varphi(x)$. Hence, to a distribution $f(x) \in \mathcal{M}_0'$ we can corespond a linearly continuous functional belonging to $\mathcal{E}'$.

If $[f_n(x)] \neq [f_n'(x)]$ are two distributions with compact supports K_1 and respectively K_2, then if $K_0 = K_1 \cup K_2$ and $\alpha(x) \in C_0^\infty(R^\nu)$ such that $\alpha(x) = 1$ on an arbitrary neighborhood V of K_0, because $\{f_n(x)\} \sim \{f_n'(x)\}$ and in usual sense, it results that

$$\lim_{n\to\infty} \int\int \dots \int_{R^\nu} f_n(x)\,(\alpha(x)\,\varphi(x))\,\mathrm{d}x \neq \lim_{n\to\infty} \int\int \dots \int_{R^\nu} f_n'(x)(\alpha(x)\,\varphi(x))\mathrm{d}x$$

as soon as supp $\alpha(x)\,\varphi(x)$ is contained in a set in which the condition required for the equivalence of two fundamental sequences is not verified. Consequently, it results that the corresponding functionals f and f' are also different in $\mathcal{E}'$.

Now let us consider $f \in \mathcal{E}'$ and $K_0 = \operatorname{supp} f$ compact. Then in any compact neighborhood V of K_0, $f = \sum_{k \leqslant p} F_k^{(k)}(x)$, where $F_k(x)$ are continuous functions with their compact supports contained in V. If $F_{nk}(x)$ are indefinitely differentiable functions such that $F_{nk}(x) \xrightarrow[u]{} F_k(x)$ and supp $F_{nk}(x) \subset \subset V$ for all $n \geqslant N(V)$, then $f_n(x) = \sum_{k \leqslant p} F_{nk}^{(k)}(x)$ is a fundamental sequence satisfying to the definition 2 and for any $\varphi(x) \in \mathcal{E}$,

$$f(\varphi) = \lim_{n\to\infty} \int\int \dots \int_{R^\nu} f_n(x)\,\varphi(x)\mathrm{d}x.$$

Finally, if we let the neighborhood V to cross the fundamental system of compact neighborhoods of K_0, then, for any V_j we obtain the sequence $\{f_{nj}(x)\}$ satisfying to the definition 2. If we take the diagonal sequence $\{f_n(x)\}$, that will satisfy to the definition 2 for any neighborhood of K_0, hence it will define a distribution.

Remark. By the proof of this theorem and because one has, in general, supp $f(x) \subset$ supp $f_n(x)$, it results that the definition 2 can be considered as a necessary and sufficient condition such that the supp $f(x)$ be the compact set K_0.

We are stoping here our general considerations on the convolution of the distributions, but in the second part of this paper we will present some problems related to the convolution equations as well as two aplications of these general results.

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ASUPRA CONVOLUŢIEI DISTRIBUŢIILOR (I)

(Rezumat)

Se prezintă unele rezultate privind convoluţia distribuţiilor în sensul lui Mikusinski [1], [6] şi echivalenţa lor cu rezultatele prezentate în [2]. În partea a doua a lucrării se vor prezenta unele aspecte referitoare la ecuaţii de convoluţie ; final se va arăta cum pot fi aplicate rezultatele generale în două cazuri concrete.

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К МЕТОДУ НАЙМЕНЬШИХ КВАДРАТОВ ДЛЯ РЕШЕНИЯ ОДНОЙ ОБОБЩЕННОЙ КРАЕВОЙ ЗАДАЧИ

М. М. КОНСТАНТИНОВ и В. А. ВЕЛЕВ

1. Рассмотрим систему

$$z'' + A(x,\lambda)\, z' + B(x,\lambda)\, z = 0, \quad x \in [a, b], \tag{1}$$

где $z = (z_1, \ldots, z_n)$, а $A = (A_{ij})_1^n$, $B = (B_{ij})_1^n$ — квадратные матрицы n-го порядка, аналитические относительно вещественного аргумента x и комплексного параметра λ. Краевая задача для отыскания собственных значений λ, порожденная системой (1) и условиями

$$z'(a) = Pz(a), \qquad z'(b) = Qz(b), \tag{2}$$

где матрицы $P = (P_{ij})_1^n$ и $Q = (Q_{ij})_1^n$ от λ не зависят, вообще говоря не может быть сведена к обычной краевой задаче.

В настоящей заметке строится приближенное решение краевой задачи (1), (2) на основе следующих предположений:

а) приближенное решение $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ имеет оскуляцию высокого порядка с точным решением z в точках $x = a$ и $x = b$;

б) невязка при подстановке нетривиального решения $\bar{z}$ в (1) — минимальна в смысле наилучшего среднеквадратического приближения.

2. При помощи (1) можно написать рекуррентные зависимости

$$z^N = C_N(x, \lambda)\, z' + D_N(x, \lambda)\, z, \quad z^{(N)} = \frac{d^N z}{dx^N}, \ (N = 0, 1, \ldots), \tag{3}$$

где матрицы $C_N = (C_{Nij})_1^n$ и $D_N = (D_{Nij})_1^n$ определяются из

$$C_{N+1} = -C_N A + \frac{\partial C_N}{\partial x} + D_N, \quad D_{N+1} = -C_N B + \frac{\partial D_N}{\partial x}, \tag{4}$$

при начальных условиях

$$(5) \qquad C_0 = 0, \quad C_1 = \delta\,; \quad D_0 = \delta, \quad D_1 = 0,$$

$(\delta = \delta_{ij})_1^n$, δ_{ij} — символ Кронекера).

Потребуем, чтобы z и $\bar{z}$ имели оскуляцию $m-1$-го порядка в точках $x = a$ и $x = b$:

$$(6) \qquad \bar{z}^{(N)}(a) = z^{(N)}(a), \qquad \bar{z}^{(N)}(b) = z^{(N)}(b), \qquad (N = 0,1,\ldots,m-1\,; \quad m \geqslant 2)$$

и положим

$$(7) \qquad \begin{cases} z(a) = (z_1(a),\ldots, z_n(a)) = u = (u_1,\ldots,u_n), \\ z(b) = (z_1(b),\ldots, z_n(b)) = v = (v_1,\ldots,v_n). \end{cases}$$

Тогда из (3), (2) и (6) получим

$$(8) \qquad \begin{cases} \bar{z}^{(N)}(a) = [C_N(a,\lambda)\, P + D_N(a,\lambda)]u, \\ \bar{z}^{(N)}(b) = [C_N(b,\lambda)\, Q + D_N(b,\lambda)]v. \end{cases}$$

Будем искать $\bar{z}$ в виде

$$(9) \qquad \bar{z}(x, u, v, \lambda) = M(u, v, \lambda)\ \varphi(x),$$

где $M = (M_{ij})_{n,2m}$ — $(n \times 2m)$-матрица, а $\varphi = (\varphi_1,\ldots, \varphi_{2m})$ — некоторая $2m$-мерная вектор функция с линейно независимыми компонентами, $m-1$ раз дифференцируемых и образующих систему чебишева на интервале $[a,b]$. Для элементов $M_{i1},\ldots, M_{i,2m}$ i-той строки матрицы M из (8) и (9) находим систему

$$(10) \qquad \begin{cases} \sum\limits_{s=1}^{2m} M_{is}\, \varphi_s^{(N)}(a) = \sum\limits_{j=1}^{n} u_j\, [D_{Nij}(a,\ \lambda) + \sum\limits_{k=1}^{n} P_{kj} C_{Nik}(a,\lambda), \\ \sum\limits_{s=1}^{2m} M_{is}\, \varphi_s^{(N)}(b) = \sum\limits_{j=1}^{n} v_j\, [D_{Nij}(b,\ \lambda) + \sum\limits_{k=1}^{n} Q_{kj}\, C_{Nik}(b,\lambda)], \\ \qquad (N = 0,\ldots,\ m-1). \end{cases}$$

Далее предполагаем, что матрица при неизвестных $M_{i1},\ldots, M_{i,2m}$ является не особой.

Решение системы (10) можно получить в виде

$$(11) \qquad \left\{ M_{is} = \sum_{j=1}^{n} [u_j\ \varepsilon_{isj}(\lambda) + v_j\ \mu_{isj}(\lambda)], \quad (i = 1,\ldots,\ n\,; \ s = 1,\ldots,\ 2m), \right.$$

где коэффициенты ε_{isj} и μ_{isj} находятся из

$$(12) \qquad \begin{cases} \sum\limits_{s=1}^{2m} \varepsilon_{isj}\ \varphi_s^{(N)}(a) = D_{Nij}(a,\lambda) + \sum\limits_{k=1}^{n} P_{kj}\, C_{Nik}(a,\lambda), \\ \sum\limits_{s=1}^{2m} \varepsilon_{isj}\ \varphi_s^{(N)}(b) = 0, \qquad (N = 0,\ldots, m-1), \end{cases}$$

и

$$(13)\quad \begin{cases} \sum\limits_{s=1}^{2m} \mu_{isj}\, \varphi_s^{(N)}(a) = 0, \\ \sum\limits_{s=1}^{2m} \mu_{isj}\, \varphi_s^{(N)}(b) = D_{Nij}\,(b,\lambda) + \sum\limits_{k=1}^{n} Q_{kj}\, C_{Nik}\,(b,\lambda), \qquad (N = 0,..,m-1). \end{cases}$$

В пространстве интегруемых с квадратом на $[a,\ b]$— n-мерных функций введем метрику в соответствии с нормой

$$(14)\qquad \|z\| = \left(\sum_{i=1}^{n} \int_a^b \omega_i\,(x)\, z_i^2(x)\, \mathrm{d}x\right)^{\frac{1}{2}},$$

где $\omega = (\omega_1,..., \omega_n)$; $\omega_1 > 0,.., \omega_n > 0$; —весовая функция.

После подстановки $\bar{z}$ в (1) получаем невязку

$$(15)\qquad H(x, u, v, \lambda) = \bar{z}'' + A\bar{z}' + B\bar{z},$$

где $H = (H_1,..., H_n)$, и функционал

$$(16)\qquad I(u, v, \lambda) = \| H \|,$$

которого надо минимизировать подходящим выбором u, v и λ, учитывая, что $|u| + |v| \neq 0$, где $|\cdot|$ — некоторая норма в n-мерном комплексном пространстве.

Ввиду того что рассматриваемая краевая задача — вполне однородная, можно принять $v_n = 1$.

Из условий

$$(17)\qquad \frac{\partial I}{\partial u_p} = 0, \quad \frac{\partial I}{\partial v_q} = 0, \quad \frac{\partial I}{\partial \lambda} = 0, \quad (p = 1,..., n\,;\ \ q = 1,..., n-1),$$

после некоторых выкладок получаем уравнения

$$(18)\quad \sum_{r=1}^{n} u_r \sum_{i=1}^{n} \int_a^b \omega_i\, G_{ir}\, G_{ip}\, \mathrm{d}x + \sum_{r=1}^{n} v_r \sum_{i=1}^{n} \int_a^b \omega_i\, H_{ir}\, G_{ip}\, \mathrm{d}x = 0, \quad (p = 1,..., n),$$

$$(19)\quad \sum_{r=1}^{n} u_r \sum_{i=1}^{n} \int_a^b \omega_i\, G_{ir}\, H_{iq}\, \mathrm{d}x + \sum_{r=1}^{n} v_r \sum_{i=1}^{n} \int_q^b \omega_i\, H_{ip}\, H_{iq}\, \mathrm{d}x = 0, \quad (q = 1,..., n-1),$$

и

$$\sum_{k=1}^{n}\sum_{r=1}^{n} u_k u_r \sum_{i=1}^{n}\int_a^b \omega_i G_{ik} \frac{\partial G_{ir}}{\partial \lambda}\, dx + \sum_{k=1}^{n}\sum_{r=1}^{n} u_k v_r \sum_{i=1}^{n}\int_a^b \omega_i \frac{\partial}{\partial \lambda}(G_{ik} H_{ir})\, dx +$$

(20)

$$+ \sum_{k=1}^{n}\sum_{r=1}^{n} v_k v_r \sum_{i=1}^{n}\int_a^b \omega_i H_{ik} \frac{\partial H_{ir}}{\partial \lambda}\, dx = 0,$$

где

$$G_{ir}(x, \lambda) = \sum_{s=1}^{2m}\sum_{j=1}^{n} \varepsilon_{jsr}(B_{ij}\varphi_s + A_{ij}\varphi_s' + \delta_{ij}\varphi_s''),$$

(21)

$$H_{ir}(x, \lambda) = \sum_{s=1}^{2m}\sum_{j=1}^{n} \mu_{jsr}(B_{ij}\varphi_s + A_{ij}\varphi_s' + \delta_{ij}\varphi_s''), \quad (i, r = 1, \ldots, n)$$

и $\varepsilon_{jsr} = \varepsilon_{jsr}(\lambda)$, $\mu_{jsr} = \mu_{jsr}(\lambda)$, $A_{ij} = A_{ij}(x, \lambda)$, $B_{ij} = B_{ij}(x, \lambda)$, $\varphi_s = \varphi_s(x)$; δ_{ij} — символ Кронекера.

Для того чтобы неоднородная система $2n-1$-го порядка (18) (19) относительно неизвестных $u_1, \ldots, u_n$, $v_1, \ldots, v_{n-1}$ имела единственное решение, необходимо и достаточно чтобы

(22) $$\operatorname{Det} R \neq 0, \qquad R = (R_{sk})_1^{2n-1},$$

где

$$R_{sk} = \begin{cases} \sum_{i=1}^{n}\int_a^b \omega_i G_{is} G_{ik}\, dx, & (1 \leqslant s \leqslant n;\ 1 \leqslant k \leqslant n), \\ \sum_{i=1}^{n}\int_a^b \omega_i G_{is} H_{i,k-n}\, dx, & (1 \leqslant s \leqslant n;\ n+1 \leqslant k \leqslant 2n-1), \\ \sum_{i=1}^{n}\int_a^b \omega_i G_{ik} H_{i,s-n}\, dx, & (n+1 \leqslant s \leqslant 2n-1;\ 1 \leqslant k \leqslant n), \\ \sum_{i=1}^{n}\int_a^b \omega_i H_{i,s-n} H_{i,k-n}\, dx, & (n+1 \leqslant s \leqslant 2n-1;\ n+1 \leqslant k \leqslant 2n-1). \end{cases}$$

При выполнении (22) из (18) и (19) можно найти неизвестные $u_1, \ldots, u_n$, $v_1, \ldots, v_{n-1}$, которые надо подставить в (20). Полученное уравнение служит для определения собственных значений λ. Заметим, что когда $A_{ij}(x, \lambda)$ и $B_{ij}(x, \lambda)$ являются полиномами от λ, уравнение (20) тоже будет полиномом от λ.

Пусть $\overline{\lambda}_1, \ldots, \overline{\lambda}_s, \ldots,$ —возрастающая последовательность приближенных собственных значений, являющихся корнями уравнения (20). Подставляя $u(\overline{\lambda}_s)$, $v(\overline{\lambda}_s)$ и $\overline{\lambda}_s$ в (9) получим приближенные зависимости для собственных функций $\overline{z}_s$:

$$\overline{z}_s\,[x, u(\overline{\lambda}_s), v(\overline{\lambda}_s), \overline{\lambda}_s] = M[u(\overline{\lambda}_s), v(\overline{\lambda}_s), \overline{\lambda}_s]\,\varphi(x), \quad (s = 1, 2, \ldots,), \tag{23}$$

где

$$\begin{cases} u(\overline{\lambda}_s) = [u_1(\overline{\lambda}), \ldots, u_n(\overline{\lambda}_s)], \\ v(\lambda_s) = [v_1(\overline{\lambda}), \ldots, v_{n-1}(\overline{\lambda}_s), 1], \end{cases} \tag{24}$$

которые соответствуют приближенным значениям $\overline{\lambda}_s$ параметра λ.

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UTILIZAREA METODEI CELOR MAI MICI PĂTRATE PENTRU REZOLVAREA UNEI PROBLEME LA LIMITĂ GENERALIZATE

(Rezumat)

Se utilizează metoda celor mai mici pătrate împreună cu unele ipoteze suplimentare, pentru construcţia soluţiei aproximative a unei probleme la limită generalizate pentru o ecuaţie liniară de ordinul doi, în spaţiul funcţiilor n-dimensionale.

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NÚCLEOS EQUIVALENTES A LA MEDIDA DIRAC

POR

ALBERTO DOU

1. Empezamos con tres lemas de carácter general que pueden aplicarse a los núcleos que aparecen en las soluciones fundamentales de ecuaciones lineales en derivadas parciales.

Lema 1. *Sea $N(0)$ un entorno cualquiera del origen de E^n. Sea $J(\zeta, s)$ una función continua para $\zeta \in E^n$ y $s \in (a, b) \subset E^1$, no negativa para $\zeta \in N(0)$ y para todo s, y tal que :*

1) *Cuando s tiende hacia a desde la derecha, $s \to a^+$, la función J tiende uniformemente hacia cero en $E^n - N(0)$.*

2) *Se tiene*

$$\lim_{s \to a^+} \int_{N(0)} J(\zeta, s)\, d\zeta = 1. \tag{1}$$

Entonces, si h es una función continua en E^n y de soporte compacto, se tiene

$$\lim_{\substack{s \to a^+ \\ \zeta \to \zeta^0}} \int_{E^n} J(\zeta - \omega, s)\, h(\omega)\, d\omega = h(\zeta^0), \tag{2}$$

cualquiera que sea $\zeta^0 \in E^n$.

Demostración. Pongamos

$$I(\zeta, s) \equiv \int_{E^n} J(\zeta - \omega, s)\, h(\omega)\, d\omega. \tag{3}$$

Sea ε positivo. Tomemos η y γ suficientemente pequeños y pongamos

$$\| \zeta - \zeta^0 \| < \eta, \quad a < s < a + \gamma. \tag{4}$$

Se tiene, llamando G al soporte compacto de h,

$$I(\zeta, s) = \int J(\zeta - \omega, s)\, h(\omega)\, d\omega = \int_{G-B} Jh\, d\omega + \int_{B} Jh\, d\omega,$$

donde $B \equiv B(\zeta^0, 2\eta)$ es la bola en E^n de centro ζ^0 y radio 2η. En virtud de la condición (1) impuesta a J, dado $\varepsilon_1 > 0$ y fijado η, existe γ positivo tal que la condición (4) implica que, para $J(\zeta - \omega, s)$ considerada como función de ω, se tenga

$$| J(\zeta - \omega, s) | < \varepsilon_1, \quad \omega \in E^n - B,$$

uniformemente para todo ζ que satisfaga la primera de las (4).

Por tanto, llamando M la medida de Lebesgue de G y poniendo

$$H \equiv \max_{\zeta \in E^n} | h(\zeta) |,$$

se tiene

$$(5) \qquad -MH\varepsilon_1 + \int_B Jh\, d\omega \leqslant I(\zeta, s) \leqslant MH\varepsilon_1 + \int_B Jh\, d\omega.$$

Por otra parte, en virtud de (1), dado ε_2 positivo existe γ positivo tal que las (4) implican

$$(6) \qquad 1 - \varepsilon_2 \leqslant \int_B J(\zeta - \omega, s)\, d\omega \leqslant 1 + \varepsilon_2,$$

uniformemente para todo ζ que satisfaga la primera de las (4).

Además, por ser h continua en ζ^0 dado ε_3 positivo existe η positivo tal que

$$(7) \qquad | h(\omega) - h(\zeta^0) | < \varepsilon_3, \quad \omega \in B.$$

Por tanto, fijamos primero η de modo que se satisfaga (7), y luego elegimos γ de modo que se satisfagan (5) y (6). Entonces, de las (5)...(7) se deduce sucesivamente

$$| I(\zeta, s) - h(\zeta^0) | \leqslant MH\varepsilon_1 + \left| \int_B J(\zeta - \omega, s)\, h(\omega)\, d\omega - h(\zeta^0) \right| \leqslant MH\varepsilon_1 +$$

$$+ H\varepsilon_2 + \left| \int_B Jh\, d\omega - h(\zeta^0) \int_B J(\zeta - \omega, s)\, d\omega \right| = MH\varepsilon_1 + H\varepsilon_2 +$$

$$+ \left| \int_B J(\zeta - \omega, s)(h(\omega) - h(\zeta^0))\, d\omega \right| \leqslant MH\varepsilon_1 + H\varepsilon_2 + \varepsilon_3 (1 + \varepsilon_2),$$

teniendo en cuenta para el último paso que J no es negativa. Por consiquiente, puesto que ε_1, ε_2 y ε_3 son arbitrarios, podemos hael último miembro menor que ε y resulta (2), c. q. d.

Las condiciones impuestas en el enunciado del lema 1 son innecesariamente restrictivas. Adjuntamos el corolario que sigue, necesario para la aplicación del lema que haremos más adelante.

Corolario al lema 1. *La conclusión del lema 1 es válida aunque s no sea escalar, sino $s \in E^n$; aunque J no sea continua, con tal que J sea localemente integrable; y aunque J no sea positiva, con tal que dado $\varepsilon_3 > 0$, exista $\delta > 0$, tal que para $|s| < \delta$ se tenga*

$$\int\limits_{|\xi - x| < \delta} |J(\xi - x, s)|\,|h(\xi) - h(x)|\,d\xi < \varepsilon_3. \tag{8}$$

Para el próximo lema introduzcamos el espacio $C_0^\alpha(E^n)$ de funciones holderianas en E^n con exponente $\alpha > 0$ y de soporte compacto. Sea $x \in E^n$, y sea $|x|$ la norma euclídea.

Lema 2. *Sea $T(x)$ continua en E^n excepto en el origen, y sea $|x|^a T(x)$ integrable en un entorno del origen.*

Si $\alpha = 0$, entonces

$$v(x) \equiv \int\limits_{E^n} T(x - \xi) f(\xi)\,d\xi \tag{9}$$

es continua, cuando f es continua y acotada en un dominio acotado $D \subset E^n$ y nula en $E^n - D$.

Si $\alpha > 0$, entonces

$$v(x) \equiv \int\limits_{|\xi - x| < 1} T(x - \xi)\,[f(\xi) - f(x)]\,d\xi \tag{10}$$

es continua cuando $f \in C_0^\alpha(E^n)$.

Demostración. Sea $B(\eta)$ la bola de centro x y radio η. Hay que demostrar que dado $\varepsilon > 0$, existe $\eta > 0$ tal que

$$|v(x^*) - v(x)| < 3\varepsilon, \quad x^* \in B(\eta).$$

Basta elegir primero γ para que la integral del integrando de (9) o (10) extendida a $B(\gamma)$ sea en valor absoluto menor que ε para todo $x^* \in B(\gamma)$. Luego elegir η, $\eta < \gamma$, para que la diferencia de integrales en x^* y en x extendidas al complementario de $B(\gamma)$ sea también menor que ε en valor absoluto.

Lema 3. *Sea I un intervalo de E^1 y sea D un dominio de E^n. Sean $x \in I$, $\xi \in D$, y $p(x, \xi)$ como función de x sea absolutamente continua en I para casi todo $\xi \in D$. Sea además $p_x \in L^1(I \times D)$. Entonces,*

$$(11) \qquad \frac{d}{dx}\int_D p(x, \xi)\, d\xi = \int_D \frac{\partial}{\partial x} p(x, \xi)\, d\xi, \text{ ct } x \in I.$$

Si además

$$\int_D p_x(x, \xi)\, d\xi$$

es continua en I, entonces vale (11) *para todo $x \in I$.*

Demostración. Basta aplicar el teorema de Fubini. Sea $I_x = I \cap (-\infty, x]$. Se tiene

$$\frac{d}{dx}\int_D p(x, \xi)\, d\xi = \frac{d}{dx}\int_D d\xi \int_{I_x} p_x(x, \xi)\, dx =$$

$$= \frac{d}{dx}\int_{I_x} dx \int_D p_x(x, \xi)\, d\xi = \int_D p_x(x, \xi)\, d\xi,$$

donde la última igualdad es ciertamente válida para ct $x \in I$; y si se cumple la condición del enunciado, entonces es válida para todo $x \in I$, c. q. d.

2. Estos lemas pueden aplicarse a numerosos núcleos que ocurren en soluciones fundamentales de ecuaciones lineales en derivadas parciales. En particular al núcleo $K(x, t)$ de la ecuación del calor y al mismo núcleo cuando se busca una solución en $E^n \times [0, T]$ que satisfaga una condición inicial para $t = 0$; también para el núcleo de Poisson cuando se quiere determinar una función armónica en una esfera y que tome valores en el contorno previamente dados.

Aquí nos limitamos como ejemplo de aplicación, al caso que sique a continuación.

Vamos a demostrar que

$$E(x - \xi, y - \eta) \equiv K(r) = -\frac{1}{2\pi}\log\frac{1}{r}, \quad r^2 = (x - \xi)^2 + (y - \eta)^2$$

es la solución fundamental de la ecuación de Laplace, $\Delta u = 0$, en el plano y análogamente se demostraría que

$$E(x - \xi) \equiv K(r) = -\frac{1}{\Omega_n}\frac{1}{r^{n-2}}, \quad r = |x - \xi|,$$

donde Ω_n es el área de la esfera unidad, es la solución fundamental de la ecuación de Laplace en E^n, para $n > 2$.

T e o r e m a 1. *Sea* $h \in C_0^\alpha(E^2)$. *Entonces, poniendo*

$$u(x, y) \equiv \int_{E^2} E(x - \xi, y - \eta)\, h(\xi, \eta)\, d\xi\, d\eta,$$

se tiene

$$u \in C^2(E^2)\,; \quad \Delta u = h(x, y), (x, y) \in E^2.$$

D e m o s t r a c i ó n. Es inmediato comprobar, mediante los lemas 2 y 3 que $u_x \in C(E^2)$ y que

$$u_x(x, y) = \int_{E^2} E_x(x - \xi, y - \eta)\, h(\xi, \eta)\, d\xi\, d\eta, \tag{12}$$

y análogamente para u_y, y ello con suponer solamente que h sea continua y acotada en un dominio acotado y nula fuera de este dominio.

Consideremos ahora u_{xx}. Se tiene

$$u_{xx}(x, y) = \frac{\partial}{\partial x} \int_{E^2} E_x(x - \xi, y - \eta)\, h(\xi, \eta)\, d\xi\, d\eta.$$

Sea $(\bar{x}, \bar{y})$ un punto fijo de E^2 y sea B el círculo de centro $(\bar{x}, \bar{y})$ y radio unidad. S. tiene

$$u_{xx}(\bar{x}, \bar{y}) = \left[\frac{\partial}{\partial x} \int_{E^2 - B} E_x h\, d\xi\, d\eta\right]_{\substack{x=\bar{x}\\ y=\bar{y}}} + h(\bar{x}, \bar{y}) \left[\frac{\partial}{\partial x} \int_B E_x(x - \xi, y - \eta)\, d\xi\, d\eta\right]_{\substack{x=\bar{x}\\ y=\bar{y}}} +$$

$$+ \left[\frac{\partial}{\partial x} \int_B E_x(x - \xi, y - \eta)\, [h(\xi, \eta) - h(\bar{x}, \bar{y})]\, d\xi\, d\eta\right]_{\substack{x=\bar{x}\\ y=\bar{y}}}.$$

Ahora bien, en este segundo miembro el primer sumando manifiestamente existe y es continuo y puede derivarse bajo el signo integral. El segundo sumando vale $\frac{1}{2}\, h(\bar{x}, \bar{y})$ según un cálculo elemental. En el tercer sumando se tiene, considerando $\bar{y}$ fijo, que

$$E_{xx}(\bar{x} - \xi, \bar{y} - \eta)\, [h(\xi, \eta) - h(\bar{x}, \bar{y})],$$

considerado como función de $(\xi, \eta) \in B$ y de $\bar{x}$ variando en un entorno I del origen, es integrable en $B \times I$ y se satisfacen obviamente todas las condiciones de los lemas 2 y 3; resulta, pues, que este último sumando existe y es continuo y se puede derivar bajo el signo integral. Resulta, pues, que $u_{xx} \in C(E^2)$. Del mismo modo se prueba que existe y es continua u_{yy}, y también u_{xy}, y en este caso el segundo sumando es nulo.

Demonstrado ya que $u \in C^2(E^2)$, consideremos Δu. Partiendo de la fórmula (12) y la análoga para u_y se tiene

$$\Delta u(x, y) = \lim_{\substack{s \to 0 \\ t \to 0}} \int_{E^n} J(x - \xi, y - \eta, s, t)\, h(\xi, \eta)\, d\xi\, d\eta,$$

siendo

$$J = J_1 + J_2,$$

$$J_1(\alpha, \beta, s) = \frac{1}{s}\left(E_x(\alpha + s, \beta) - E_x(\alpha, \beta)\right) = \frac{1}{2\pi s}\left(\frac{\alpha + s}{(\alpha + s)^2 + \beta^2} - \frac{\alpha}{\alpha^2 + \beta^2}\right),$$

$$J_2(\alpha, \beta, t) = \frac{1}{2\pi t}\left(\frac{\beta + t}{\alpha^2 + (\beta + t)^2} - \frac{\beta}{\alpha^2 + \beta^2}\right).$$

Se comprueba que J satisface las condiciones del lema 1 o las del corolario. En particular, si B es un entorno circular del origen con centro en el mismo, se tiene

$$\lim_{s \to 0} \int_{E^2} J_1(\alpha, \beta, s)\, d\alpha\, d\beta =$$

$$= -\frac{1}{2\pi} \int_{\partial B} \lim_{s \to 0} \frac{1}{s}\left(\operatorname{Arc\,tg} \frac{\beta}{-\alpha + s} - \operatorname{Arc\,tg} \frac{\beta}{-\alpha}\right) d\alpha =$$

$$= \frac{1}{2\pi} \int_{\partial B} \frac{\partial \theta}{\partial \alpha}, \qquad \frac{\beta}{\alpha}.$$

O sea

$$\lim_{\substack{s \to 0 \\ t \to 0}} \int_B J(\alpha, \beta, s, t)\, d\alpha\, d\beta = \frac{1}{2\pi} \int_{\partial B} \left(\frac{\partial \theta}{\partial \alpha} d\alpha + \frac{\partial \theta}{\partial \beta} d\beta\right) = 1.$$

También se comprueba que para $h \in C_0^\alpha(E^2)$ se cumple la condición (8). Por tanto, el lema 1 arroja

$$\Delta u = h(x, y), \qquad (x, y) \in E^2,$$

c. q. d.

3. El resultado del teorema 1 resulta más simple en teoría de distribuciones. Se define la solución fundamental $E(x - \xi)$ mediante

$$\Delta E(x - y) = \delta(x - \xi),$$

y empleando el producto de convolución se obtiene

$$\Delta \int_{E^n} E(x-\xi)\, h(\xi)\, d\xi = (\Delta(E^* h))(x) = ((\Delta E)^* h)(x) = (\delta^* h)(x) = h(x).$$

Esta demostración es válida para toda función continua en E^n y de soporte compacto, sin que sea necesario suponer que sea holderiana. Por otra parte el teorema 1 puede ser falso para h continua. Véase un ejemplo en B. Epstein [1]. El diverso resultado de las dos teorías es debido a la diferente definición del concepto de derivada. Así, en la teoría de distribuciones, aunque no puede escribirse, para la derivada segunda de $\log r$ con $n = 2$,

$$\frac{\partial}{\partial x} \frac{x-\xi}{r^2} = \frac{(x-\xi)^2 - (y-\eta)^2}{r^4},$$

porque el segundo miembro es una función que no es localmente integrable, y por tanto no puede identificarse con una distribución, resulta, en cambio que se tiene

$$\frac{\partial}{\partial x} \frac{x-\xi}{r^2} = \mathrm{vp} \frac{(x-\xi)^2 - (y-\eta)^2}{r^4},$$

o sea, de una manera más explícita,

$$\left(\frac{\partial}{\partial x} \frac{x-\xi}{r^2}\right)(\varphi) = \mathrm{vp} \iint_{E^2} \frac{(x-\xi)^2 - (y-\eta)^2}{r^4} \varphi(x, y)\, dx\, dy \equiv$$

$$\equiv \lim_{\varepsilon \to 0} \iint_{E-B} \frac{(x-\xi)^2 - (y-\eta)^2}{r^4} \varphi(x, y)\, dx\, dy, \quad \varphi \in C_0^\infty(E^2),$$

donde B es el círculo de centro (ξ, η) y radio ε.

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NUCLEE ECHIVALENTE CU MĂSURA DIRAC

(Rezumat)

Se demonstrează trei leme generale privitoare la continuitatea şi derivabilitatea integralelor, care se aplică la cazul particular al nucleului lui Poisson, regăsindu-se astfel în mod sistematic proprietăţile sale clasice. Rezultatul obţinut este comparat cu cel corespunzător din teoria distribuţiilor.

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^n} \Delta(x - \xi)\, \lambda(\xi)\, d\xi = (\Delta(\xi^* \lambda))(x) = ((\Delta\lambda)^* \lambda)(x) = (\delta^* \lambda)(x) = \lambda(x).$$

Esta demostración es válida para toda función continua en $\mathbb{R}^n$ y de soporte compacto, sin que sea necesario suponer que sea indefinida. Por otra parte el teorema 1 puede ser falso para λ continua. Véase un ejemplo en B. Hypsleiro [1]. El diverso resultado de las dos teorías es debido a la diferente definición del concepto de derivada. Así, en la teoría de distribuciones, aunque no puede escribirse, para la derivada segunda de $\log r$ (con $r = \xi$),

$$\frac{d^2 \log r}{dr^2} = \frac{(x - \xi)^2 - (y - \xi)^2}{r^2},$$

porque el segundo miembro es una función que no es localmente integrable, y por tanto no puede identificarse con una distribución, resulta, en cambio que se tiene

$$\frac{d^2 \log r}{dx^2} = \mathrm{vp}\, \frac{(x - \xi)^2 - (y - \xi)^2}{r^2},$$

o sea, de una manera más explícita,

$$\left(\frac{d^2 \log r}{dx^2}\right)(\varphi) = \mathrm{vp} \int_{\mathbb{R}^2} \frac{(x - \xi)^2 - (y - \xi)^2}{r^2}\, \varphi(x, y)\, dx\, dy =$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 - B} \frac{(x - \xi)^2 - (y - \xi)^2}{r^2}\, \varphi(x, y)\, dx\, dy, \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2),$$

donde B es el círculo de centro (ξ, η) y radio ε.

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NUCLEOS EQUIVALENTES DE MATRICES DIRAC

(Resumen)

Se demuestran tres bonitas generales propiedades de equivalencia y derivabilidad interpretables, cada una de ellas, en cierta particular al problema del Dirac, especialmente útil en muchas aplicaciones propiedades que citadas. Resultados obtenidos este compuesto en del correspondiente dos teorías distributivas.

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О НЕКОТОРЫХ НЕЛИНЕЙНЬХ КРАЕВЫХ ЗАДАЧАХ (II)

И. Т. КИГУРАДЗЕ

1. *Введение*

В первыи части этои работы [1] был рассмотрен некоторые определения и вспомогательные приложения необходимые для исследования задачи счществования решения $u_1(t)$, $u_2(t)$ системы обыкновенных дифференциальных уравнений

$$\frac{\mathrm{d}u_k}{\mathrm{d}t} = f_k(t, u_1, u_2), \qquad (k = 1,2), \tag{1.1}$$

определенного в промежутке $a < t < b$ и удовлетворяющего одному из следующих двух краевых условий

$$\Phi_1[u_1(a),\ u_2(a)] = 0, \qquad \Phi_2[u_1(b),\ u_2(b)] = 0\,; \tag{1.2}$$

$$u_2(a) = \varphi\,[u_1(a)], \quad \sup_{a<t<b} |u_1(t)| < +\infty. \tag{1.3}$$

В настоящеи работе рассматривается достаточные условия разрешимости задачи (1.1) — (1.2) и задачи (1.1) — (1.3).

2. *Достаточные условия разрешимости задачи* (1.1) — (1.2)

Всюду в этом параграфе предполагается, что

$$-\infty < a < b < +\infty, \quad f_j(t,\ x,\ y) \in K_t(a,\ b), \quad (j = 1,2) \tag{2.1}$$

и

$$\begin{gathered} \Phi_1\,[\varphi_1(x, y),\ \varphi_2(x, y)] \equiv 0, \ \Phi_2\,[x + \varphi_1(x, y),\ y + \varphi_2(x, y)] \equiv 0 \\ \text{при } -\infty < x, y < +\infty, \end{gathered} \tag{2.2}$$

где $\varphi_j(x, y)$, $(j = 1,2)$ — непрерывные в области $-\infty < x, y < +\infty$ функции.

Теорема 2.1. *Если*

$$\sum_{k=1}^{2} |f_k(t, x, y)| \leqslant \psi(t) \quad \text{при} \quad a \leqslant t \leqslant b, \quad -\infty < x, y < +\infty, \tag{2.3}$$

где

$$\psi(t) \in L(a, b), \tag{2.4}$$

то задача (1.1) — (1.2) *имеет, по крайней мере, одно решение.*

Доказательство. Положим

$$\max_{|x| + |y| < \int_a^b \Psi(t) dt} \sum_{k=1}^{2} |\varphi_k(x, y)| = M. \tag{2.5}$$

Пусть $C^2_{[a,b]}$ — пространство непрерывных на отрезке $[a, b]$ двумерных вектор-функций $\vec{u}(t) = \{u_1(t), u_2(t)\}$ с нормой

$$\|\vec{u}(t)\| = \max_{a \leqslant t \leqslant b} (|u_1(t)| + |u_2(t)|), \tag{2.6}$$

а S — множество из этого пространства, определенное следующим образом

$$\vec{u}(t) = \{u_1(t), u_2(t)\} \in S, \quad \text{если} \quad \|\vec{u}(t)\| \leqslant M + \int_a^b \psi(t)\, dt. \tag{2.7}$$

Рассмотрим оператор

$$A\vec{u}(t) = \{A_1\vec{u}(t), \ A_2\vec{u}(t)\}, \tag{2.8}$$

где

$$\begin{aligned} A_k\vec{u}(t) = \varphi_k\left[\int_a^b f_1(\tau, u_1(\tau), u_2(\tau))\, d\tau, \int_a^b f_2(\tau, u_1(\tau), u_2(\tau))\, d\tau\right] + \\ + \int_a^t f_k(\tau, u_1(\tau), u_2(\tau))\, d\tau, \qquad (k = 1,2). \end{aligned} \tag{2.9}$$

Из (2.3)...(2.9) очевидно, что если $\vec{u}(t) \in S$, то

$$A\vec{u}(t) \in S \text{ и } \sum_{k=1}^{2} |A_k\vec{u}(t_1) - A_k\vec{u}(t_2)| \leqslant \left|\int_{t_1}^{t_2} \psi(\tau)\, d\tau\right| \text{ при любых } t_1, t_2 \in [a, b].$$

Отсюда согласно лемме Арцелла-Асколи, ясно, что AS является компактным множеством пространства $C^2_{[a,b]}$. С другой стороны, в силу (2.1), нетрудно проверить, что A является непрерывным оператором. Следовательно, непрерывный оператор A преобразует шар S в свое же компактное подмножество. Поэтому, согласно теореме Шаудера (см. напр. [2]), найдется такая функция $\vec{u}(t) = \{u_1(t), u_2(t)\} \in S$, что $A\vec{u}(t) = \vec{u}(t)$ при $a \leqslant t \leqslant b$, т.е.

$$u_k(t) = \varphi_k \left[\int_a^b f_1(\tau, u_1(\tau), u_2(\tau))\, d\tau, \int_a^b f_2(\tau, u_1(\tau), u_2(\tau))\, d\tau \right] + \tag{2.10}$$
$$+ \int_a^t f_k(\tau, u_1(\tau), u_2(\tau))\, d\tau \quad \text{при} \quad a \leqslant t \leqslant b, \quad (k = 1,2).$$

Отсюда, в силу (2.2), очевидно, что $u_1(t), u_2(t)$ является решением задачи (1.1) — (1.2).

Замечание. Фактически, нами доказано существование такого решения $u_1(t)$, $u_2(t)$ задачи (1.1) — (1.2), которое удовлетворяет и системе интегральных уравнений (2.10).

Теорема 2.2. *Пусть*

$$\sigma_k(t), \sigma_k'(t), (k = 1, 2) \text{ — абсолютно непрерывны и } \sigma_1(t) \leqslant \sigma_2(t) \text{ при } a \leqslant t \leqslant b; \tag{2.11}$$

$$\Phi_1(x, y) \neq 0 \text{ при } x < \sigma_1(a),\ y > \sigma_1'(a) \text{ и при } x > \sigma_2(a),\ y < \sigma_2'(a);$$
$$\Phi_2(x, y) \neq 0 \text{ при } x < \sigma_1(b),\ y \leqslant \sigma_1'(b) \text{ и при } x > \sigma_2(b),\ y \geqslant \sigma_2'(b); \tag{2.12}$$

$$[f_1(t, x, y) - \sigma_k'(t)] \operatorname{sign} [y - \sigma_k'(t)] > 0 \text{ при } a < t < b,\ \sigma_1(t) \leqslant x \leqslant \sigma_2(t),$$
$$y \neq \sigma_k'(t),\ (k = 1,2); \tag{2.13}$$

$$(-1)^k [f_2(t, \sigma_k(t), \sigma_k'(t)) - \sigma_k''(t)] \geqslant 0 \text{ при } a < t < b,\ (k = 1,2) \text{ и} \tag{2.14}$$

$$\sum_{k=1}^{2} |f_k(t, x, y)| \leqslant \psi(t) \text{ при } a < t < b,\ \sigma_1(t) - \varepsilon \leqslant x \leqslant \sigma_2(t) + \varepsilon, \tag{2.15}$$

где $\varepsilon > 0$, *а* $\psi(t)$ *удовлетворяет условию* (2.4). *Тогда задача* (1.1) — (1.2) *имеет, по крайней мере, одно решение* $u_1(t)$, $u_2(t)$ *такое, что*

$$\sigma_1(t) \leqslant u_1(t) \leqslant \sigma_2(t) \quad \text{при} \quad a \leqslant t \leqslant b. \tag{2.16}$$

Доказательство. Рассмотрим систему дифференциальных уравнений

$$\frac{du_k}{dt} = f_{kj}(t, u_1, u_2), \quad (k = 1,2), \tag{2.17}$$

где

$$(2.18)\qquad f_{1j}(t,\ x,\ y)=\begin{cases} f_1(t,\ \sigma_1(t),y) & \text{при}\quad x<\sigma_1(t)\\ f_1(t,x,y) & \text{при}\quad \sigma_1(t)\leqslant x\leqslant\sigma_2(t)\\ f_1(t,\ \sigma_2(t),\ y) & \text{при}\quad x>\sigma_2(t),\end{cases}$$

$$(2.19)\qquad f_{2j}(t,x,y)=\begin{cases} f_2(t,\sigma_1(t),\ \sigma_1'(t)) & \text{при}\quad x<\sigma_1(t)-\dfrac{\varepsilon}{j}\\ \dfrac{j}{\varepsilon}\left(x-\sigma_1(t)+\dfrac{\varepsilon}{j}\right)f_2(t,\ \sigma_1(t),\ y)+ & \\ +\dfrac{j}{\varepsilon}(\sigma_1(t)-x)f_2(t,\sigma_1(t),\sigma_1'(t)) & \text{при}\quad \sigma_1(t)-\dfrac{\varepsilon}{j}\leqslant x<\sigma_1(t)\\ f_2(t,x,y) & \text{при}\quad \sigma_1(t)\leqslant x\leqslant\sigma_2(t)\\ \dfrac{j}{\varepsilon}\left(\sigma_2(t)+\dfrac{\varepsilon}{j}-x\right)f_2(t,\sigma_2(t),y)+ & \\ +\dfrac{j}{\varepsilon}(x-\sigma_2(t))f_2(t,\sigma_2(t),\sigma_1'(t)) & \text{при}\quad \sigma_2(t)<x\leqslant\sigma_2(t)+\dfrac{\varepsilon}{j}\\ f_2(t,\sigma_2(t),\sigma_2'(t)) & \text{при}\quad x>\sigma_2(t)+\dfrac{\varepsilon}{j}.\end{cases}$$

В силу (2.15), (2.18) и (2.19), легко проверить, что при любом натуральном j система дифференциальных уравнений (2.17) удовлетворяет всем условиям теоремы 2.1. Поэтому, согласно замечанию к теореме 2.1, система (2.17) имеет решение $u_{1j}(t)$, $u_{2j}(t)$, такое, что

$$(2.20)\qquad \Phi_1[u_{1j}(a),u_{2j}(a)]=0,\quad \Phi_2[u_{1j}(b),u_{2j}(b)]=0$$

и

$$(2.21)\qquad \begin{aligned} u_{kj}(t)=\varphi_k\Big[\int_a^b f_{ij}(\tau,u_{1j}(\tau),u_{2j}(\tau))\,d\tau,\quad \int_a^b f_{2j}(\tau,\ u_{1j}(\tau),\ u_{2j}(\tau))\,d\tau\Big]+\\ +\int_a^t f_{kj}(\tau,u_{1j}(\tau),u_{2j}(\tau))\,d\tau\quad \text{при}\quad a\leqslant t\leqslant b,\ (k=1,2).\end{aligned}$$

Покажем, что

$$(2.22)\qquad \sigma_1(t)-\frac{\varepsilon}{j}\leqslant u_{1j}(t)\leqslant\sigma_2(t)+\frac{\varepsilon}{j}\quad \text{при}\quad a\leqslant t\leqslant b.$$

Допустим противное. Тогда найдутся такие числа t_1 и t_2, $a\leqslant t_1<t_2\leqslant b$, что

$$(2.23)\quad (-1)^k\left[u_{1j}(t)-\sigma_k(t)-\frac{(-1)^k\varepsilon}{j}\right]>0 \text{ при } t_1<t<t_2, \text{ где } k=1 \text{ или } 2.$$

При этом, без ограничения общности, можем считать, что если

$$(2.24)\quad a<t_i<b, \text{ где } i=1 \text{ или } 2, \text{ то } u_{1j}(t_i)-\sigma_k(t_i)-\frac{(-1)^k\varepsilon}{j}=0.$$

Согласно (2.14), (2.19) и (2.23), будем иметь

$$(-1^k[u_{2j}(t)-\sigma_k'(t)]'=(-1)^k[f_2(t,\ \sigma_k(t),\ \sigma_k'(t))-\sigma_k''(t)]\geqslant 0 \text{ при } t_1\leqslant t\leqslant t_2.$$

Отсюда очевидно, что либо

$$(2.25)\qquad (-1)^k[u_{2j}(t)-\sigma_k'(t)]<0 \quad \text{при} \quad t_1\leqslant t<t_2,$$

либо

$$(2.26)\quad (-1)^k[u_{2j}(t)-\sigma_k'(t)]\geqslant 0 \quad \text{при} \quad t^*\leqslant t\leqslant t_2, \quad \text{где} \quad t_1\leqslant t^*<t_2.$$

Предположим сперва, что соблюдается условие (2.25). Тогда, в силу (2.13) и (2.18), будем иметь

$$(-1)^k\left[u_{1j}(t)-\sigma_k(t)-\frac{(-1)^k\varepsilon}{j}\right]'<0 \quad \text{при} \quad t_1\leqslant t<t_2,$$

согласно чему из (2.23) следует, что

$$(2.27)\qquad (-1)^k\left[u_{1j}(t_1)-\sigma_k(t_1)-\frac{(-1)^k\varepsilon}{j}\right]>0.$$

Из (2.24), (2.25) и (2.27) ясно, что $t_1=a$,

$$(-1)^k[u_{1j}(a)-\sigma_k(a)]>0, \quad (-1)^k[u_{2j}(a)-\sigma_k'(a)]<0.$$

Поэтому, согласно (2.12), имеем $\Phi_1[u_{1j}(a),\ u_{2j}(a)]\neq 0$, что противоречит условию (2.20). Следовательно, неравенство (2.25) не может иметь места. Совершенно аналогично покажем, что не имеет места неравенство (2.26), что и доказывает справедливость оценки (2.22).

В силу (2.5), (2.15), (2.18) и (2.19), из (2.21) имеем

$$\sum_{k=1}^{2}|u_{kj}(t)|\leqslant M+\int_a^b\psi(\tau)\,d\tau,\quad \sum_{k=1}^{2}|u_{kj}(t_2)-u_{kj}(t_1)|\leqslant\left|\int_{t_1}^{t_2}\psi(\tau)\,d\tau\right|,$$

$$\text{при} \quad t_1,\ t_2\in[a,b], \quad (j=1,2\ldots).$$

Следовательно, $\{u_{1j}(t)\}$ и $\{u_{2j}(t)\}$ являются последовательностями равномерно ограниченных и равностепенно непрерывных в промежутке $[a,\ b]$ функций. Поэтому, согласно лемме Арцелла-Асколи, без ограничения общности может считать, что они равномерно сходятся. Положим

$$\lim_{j\to\infty} u_{kj}(t) = u_k(t), \quad (k = 1,2). \tag{2.28}$$

В силу (2.18), (2.22) и (2.28) ясно, что соблюдается условие (2.16) и

$$\lim_{j\to\infty} f_{1j}(t, u_{1j}(t), u_{2j}(t)) = f_1(t, u_1(t), u_2(t)) \quad \text{при} \quad a < t < b. \tag{2.29}$$

Докажем, что

$$\lim_{j\to\infty} f_{2j}((t, u_{1j}(t), u_{2j}(t)) = f_2(t, u_1(t), u_2(t)) \quad \text{при} \quad a < t < b. \tag{2.30}$$

Согласно (2.19), (2.22) и (2.28), для этого достаточно показать, что если

$$u_1(t_0) - \sigma_k(t_0) = 0, \tag{2.31}$$

где $a < t_0 < b$, $k = 1$ или 2, то $u_2(t_0) - \sigma_k'(t_0) = 0$.

Допустим противное, пусть для некоторого $t_0 \in (a, b)$ соблюдается равенство (2.31) и

$$u_2(t) - \sigma_k'(t) \neq 0 \quad \text{при} \quad t_0 - \delta < t < t_0 + \delta,$$

где $\delta > 0$. Тогда, в виду (2.13), (2.16) и (2.29), будем иметь

$$[u_1(t) - \sigma_k(t)]' \neq 0 \quad \text{при} \quad t_0 - \sigma < t < t_0 + \delta.$$

Отсюда, в виду (2.31), следует, что функци $u_1(t) - \sigma_k(t)$ меняет знак, что противоречит условию (2.16). Полученное противоречие доказывает, что соблюдается условие (2.30).

Перейдя к пределу в равенствах (2.21), когда $j \to \infty$ согласно (2.28), (2.29) и (2.30), заключим, что соблюдаются равенства (2.10), т.е. $u_1(t), u_2(t)$ является решением задачи (1.1) — (1.2). Теорема доказана.

Теорема 2.3. *Пусть соблюдаются условия* (2.11), (2.12), (2.13) *и* (2.14), *а в области* $\sigma_1(t) \leqslant x \leqslant \sigma_2(t)$, $-\infty < y < +\infty$ *имеем*

$$f_1(t, x, y) \operatorname{sign} y \geqslant \omega(t, |y_1|) \quad \textit{при} \quad a < t < b; \tag{2.32}$$

$$\begin{aligned} &f_2(t, x, y) \operatorname{sign} y \geqslant -\omega_1(t, |y|) \ \textit{при} \ a \leqslant t < \beta; \\ &f_2(t, x, y) \operatorname{sign} y \leqslant \omega_2(t, |y|) \quad \textit{при} \quad \alpha < t \leqslant b, \end{aligned} \tag{2.33}$$

где

$$\omega(t, y) \ \textit{не убывает по } y, \ \lim_{y\to+\infty} \int_a^b \omega(t, y)\, dt > r; \tag{2.34}$$

$$\{\omega(t, y), \omega_1(t, y)\} \in B_1(r; a, \beta_1, \beta); \{\omega(t, y), \omega_2(t, y)\} \in B_2(r; \alpha, \alpha_1, b), \tag{2.35}$$

$$a \leqslant \alpha < \alpha_1 \leqslant \beta_1 < \beta \leqslant b; \ r = \max_{a \leqslant t, \tau \leqslant b} |\sigma_1(t) - \sigma_2(\tau)|. \tag{2.36}$$

Тогда задача (1.1)—(1.2) *имеет, по крайней мере, одно решение* $u_1(t)$, $u_2(t)$, *удовлетворяющее условию* (2.16).

Доказательство. Согласно [1, определениям 1.2 и 1.3], из (2.35) и (2.36) легко заключим, что найдется такое положительное число ρ, что какие бы ни были абсолютно непрерывные в промежутке $[a, b]$ функции $u_1(t)$ и $u_2(t)$, если соблюдается неравенство (2.16) и

$$
\begin{aligned}
&u_1'(t)\ \operatorname{sign} u_2(t) \geqslant \omega(t, |u_2(t)|) \text{ при } a \leqslant t \leqslant b,\\
(2.37)\qquad &u_2'(t)\ \operatorname{sign} u_2(t) \geqslant -\omega_1(t, |u_2(t)|) \text{ при } a \leqslant t \leqslant \beta,\\
&u_2'(t)\ \operatorname{sign} u_2(t) \leqslant \omega_2(t, |u_2(t)|) \text{ при } \alpha \leqslant t \leqslant b,
\end{aligned}
$$

то будем иметь

$$
(2.38)\qquad |u_2(t)| \leqslant \rho \text{ при } a \leqslant t \leqslant b.
$$

Согласно (2.34) найдется такое положительное число y_0 что

$$
(2.39)\qquad \int_a^b \omega(t, y_0)\, \mathrm{d}t > r.
$$

Положим

$$
(2.40)\qquad \rho_2 = \rho + \max_{a \leqslant t \leqslant b} \sum_{k=1}^{2} (|\sigma_k(t)| + |\sigma_k'(t)|),
$$

$$
(2.41)\qquad h(t) = \max_{|x|+|y| \leqslant 2\rho_2} |f_2(t, x, y)|,
$$

$$
(2.42)\qquad \rho_1 = y_0 + \rho_2 + \int_a^b h(\tau)\, \mathrm{d}\tau.
$$

Рассмотрим систему

$$
(2.43)\qquad \frac{\mathrm{d}u_k}{\mathrm{d}t} = g_k(t, u_1, u_2), \qquad (k = 1, 2),
$$

где

$$
(2.44)\qquad g_1(t, x, y) = \begin{cases} f_1(t, x, \rho_1) \text{ при } y < -\rho_1 \\ f_1(t, x, y) \text{ при } -\rho_1 \leqslant y \leqslant \rho_1 \\ f_1(t, x, \rho_1) \text{ при } y \geqslant \rho_1; \end{cases}
$$

$$(2.45)\qquad g_2(t, x, y) = \begin{cases} f_2(t, x, y) \text{ при } |y| < \rho_2 \\ 2\left(1 - \dfrac{|y|}{2\rho_2}\right) \text{ при } \rho_2 \leqslant |y| \leqslant 2\rho_2 \\ 0 \text{ при } |y| > 2\rho_2. \end{cases}$$

Нетрудно проверить, что система (2.43) удовлетворяет всем условиям теоремы 2.2, поэтому она имеет решение $u_1(t)$, $u_2(t)$ удовлетворяющее условиям (1.2) и (2.16). Наша задача доказать, что $u_1(t)$, $u_2(t)$ является решением системы (1.1)

Докажем, прежде всего, что

$$(2.46)\qquad |u_2(t_0)| \leqslant y_0 \quad \text{для некоторого} \quad t_0 \in [a, b].$$

Допустим противное, что $|u_2(t)| > y_0$ при $a \leqslant t \leqslant b$. Тогда, согласно (2.32), (2.34), (2.42) и (2.44), будем иметь

$$u_1'(t) \operatorname{sign} u_2(a) \geqslant \omega(t, y_0) \quad \text{при} \quad a \leqslant t \leqslant b.$$

Отсюда, в силу (2.16) и (2.36), следует

$$\int_a^b \omega(t, y_0)\, dt \leqslant |u_1(a) - u_1(b)| \leqslant r,$$

что противоречит неревенству (2.39). Полученное противоречие доказывает справедливость условия (2.46).

В силу (2.16), (2.40)... (2.42), (2.45) и (2.46), из равенства

$$u_2(t) = u_2(t_0) + \int_{t_0}^{t} g_2(\tau, u_1, u_2(\tau))\, d\tau$$

получим

$$(2.47)\qquad |u_2(t)| \leqslant \rho_1 \quad \text{при} \quad a \leqslant t \leqslant b.$$

В виду (2.44) и (2.47), ясно, что

$$(2.48)\qquad u_1'(t) = f_1(t, u_1(t), u_2(t)) \quad \text{при} \quad a \leqslant t \leqslant b.$$

Поскольку $\omega_1(t, y)$ и $\omega_2(t, y)$ неотрицательны, согласно (2.32), (2.33), (2.45) и (2.48), очевидно, что соблюдаются неравенства (2.37), откуда, как было отмечено выше, следует неравенство (2.38).

В силу (2.38), (2.40) и (2.45), имеем

$$u_2'(t) = f_2(t, u_1(t), u_2(t)) \quad \text{при} \quad a \leqslant t \leqslant b.$$

Следовательно, $u_1(t)$, $u_2(t)$ является решением системы (1). Теорема доказана.

Согласно [1, леммам 1.1, 1.2, 1.1′ и 1.2′] и их следствиям, из теоремы 2.3 легко вытекают следующие предложения:

Следствие 1. *Пусть соблюдаются условия* (2.11), (2.12), (2.13) и (2.14) *а в области* $\sigma_1(t) \leqslant x \leqslant \sigma_2(t)$, $-\infty < y < +\infty$ *имеют место неравенства* (2.32) *и* (2.33), *где* $a \leqslant \alpha < \beta \leqslant b$; $\omega(t, y)$ *определена и неотрицательна в области* $a \leqslant t \leqslant b$, $0 \leqslant y < +\infty$; $\omega(t, |y|) \in K_t(a, b)$ *и*

$$\lim_{y \to +\infty} \int_{\alpha}^{\beta} \omega(t, y)\, dt = +\infty.$$

Тогда для разрешимости задачи (1.1)—(1.2) *достаточно, чтобы функция* $\omega_1(t, y)$ *удовлетворяла одному из следующих двух условий*:

1) $\omega_1(t, y)$ *опрееделена и неотрицательна в области* $a \leqslant t \leqslant b$, $0 \leqslant y < +\infty$; $\omega_1(t, |y|) \in K_t(a, b)$ *и при любом* $\rho_0 \in (0, +\infty)$ *верхнее решение* $\rho(t)$ *задачи* $d\rho/dt = -\omega_1(t, \rho)$, $\rho(\beta) = \rho_0$ *определено и ограничено в промежутке* $[a, \beta]$;

2) $\omega_1(t, y) = \omega_0(y) \sum_{k=1}^{m} \psi_k(t) [\omega(t, y)]^{\frac{1}{q_k}}$,

где

$$\psi_k(t) \geqslant 0,\ \psi_k(t) \in L^{p_k}(a, b),\ \frac{1}{p_k} + \frac{1}{q_k} = 1,\ 1 \leqslant q_k \leqslant +\infty \tag{2.49}$$

$$(k = 1, 2, \ldots, m),$$

$$\omega_0(t) > 0 \text{ непрерывна при } 0 \leqslant t < +\infty \text{ и } \int_0^{+\infty} \frac{dt}{\omega_0(t)} = +\infty,$$

а функция $\omega_2(t, y)$ — *одному из следующих двух условий*:

1) $\omega_2(t, y)$ *определена и неотрицательна в области* $a \leqslant t \leqslant b$, $0 \leqslant y < +\infty$; $\omega_2(t, |y|) \in K_t(a, b)$ *и при любом* $\rho_0 \in (0, +\infty)$ *верхнее решение* $\rho(t)$ *задачи* $d\rho/dt = \omega_2(t, \rho)$, $\rho(\alpha) = \rho_0$ *определено и ограничено в промежутке* $[\alpha, b]$;

2) $\omega_2(t, y) = \omega_0(y) \sum_{k=1}^{m} \psi_k(t) [\omega(t, y)]^{\frac{1}{q_k}}$, *где* $\omega_0(t)$, $\psi_k(t)$ *и* q_k, $(k=1, 2, \ldots, m)$ *удовлетворяют условиям* (2.49).

Следствие 2. *Пусть соблюдаются условия* (2.11), (2.12), (2.13) и (2.14), *а в области* $\sigma_1(t) \leqslant x \leqslant \sigma_2(t)$, $-\infty < y < +\infty$ *имеем*

$$f_1(t, x, y) \operatorname{sign} y \geqslant \psi(t) |y|^{\lambda} \text{ при } a \leqslant t \leqslant b;$$

$$f_2(t, x, y) \operatorname{sign} y \geqslant -\psi_0(t) |y|^{\lambda_0} \text{ при } a \leqslant t \leqslant \alpha;$$

$$f_2(t, x, y) \operatorname{sign} y \leqslant (1 + |y|) \sum_{k=1}^{m} \psi_k(t) |y|^{\lambda_k} \text{ при } a < t \leqslant b,$$

где

$$a < \alpha < b, \quad 0 < \lambda + \infty, \quad 1 < \lambda_0 < +\infty, \quad 0 \leqslant \lambda_k \leqslant \lambda, \quad (k = 1, 2, \ldots, m),$$

$$\psi(t) > 0, \quad \psi_0(t) > 0, \quad \psi(t) \in L(a, b), \quad \psi_0(t) \in L(a, b),$$

$$\int_a^\alpha \psi(t) \left[\int_a^t \psi_0(\tau) d\tau \right]^{\frac{\lambda}{1-\lambda_0}} dt = +\infty$$

и

$$\psi_k(t)[\psi(t)]^{-\frac{\lambda_k}{\lambda}} \in L^{p_k}(t_0, b) \text{ при любом } t_0 \in (a, b), \; p_k = \frac{\lambda}{\lambda - \lambda_k}, \; (k = 1, 2, \ldots, m).$$

Тогда задача (1.1) — (1.2) *имеет, по крайней мере, одно решение.*

Следствие 3. *Пусть соблюдаются условия* (2.11), (2.12), (2.13) *и* (2.14), *а в области* $\sigma_1(t) \leqslant x \leqslant \sigma_2(t)$, $-\infty < y < +\infty$ *имеем*

$$f_1(t, x, y) \operatorname{sign} y \geqslant \psi(t) |y|^\lambda \text{ при } a \leqslant t \leqslant b;$$

$$f_2(t, x, y) \operatorname{sign} y \geqslant -(1 + |y|) \sum_{k=1}^m \psi_k(t) |y|^{\lambda_k} \text{ при } a \leqslant t < b;$$

$$f_2(t, x, y) \operatorname{sign} y \leqslant \psi_0(t) |y|^{\lambda_0} \text{ при } \alpha \leqslant t \leqslant b,$$

где

$$a < \alpha < b, \; 0 < \lambda < +\infty, \; 1 < \lambda_0 < +\infty, \; 0 \leqslant \lambda_k \leqslant \lambda, \; (k = 1, 2, \ldots, m),$$

$$\psi(t) > 0, \quad \psi_0(t) > 0, \quad \psi(t) \in L(a, b), \quad \psi_0(t) \in L(a, b),$$

$$\int_\alpha^b \psi(t) \left[\int_t^b \psi_0(\tau) d\tau \right]^{\frac{\lambda}{1-\lambda_0}} dt = +\infty$$

и

$$\psi_k(t)[\psi(t)]^{\frac{-\lambda}{}} \in L^{p_k}(a, t_0) \text{ при любом } t_0 \in (a, b), \; p_k = \frac{\lambda}{\lambda - \lambda_k}, \; (k = 1, 2, \ldots, m).$$

Тогда задача (1.1)—(1.2) *имеет, по крайней мере, одно решение.*

Полагая $f_1(t, x, y) = y$, $f_2(t, x, y) = f(t, x, y)$ и $\Phi_k(x, y) = x - l_k y - c_k$, $(k = 1, 2)$, согласно следствию 1, легко проверить, что имеет место такое

Следствие 4. *Если* $-\infty < \sigma_1 < \sigma_2 < +\infty$; $0 < l_1 + \infty$; $-\infty < l_2 \leqslant 0$, $\sigma_1 \leqslant c_1$, $c_2 \leqslant \sigma_2$;

$$(-1)^k f(t, \sigma_k, 0) \geqslant 0 \text{ при } a < t < b, \quad (k = 1, 2), \tag{2.50}$$

а в области $\sigma_1 \leqslant x \leqslant \sigma_2$, $-\infty < y < +\infty$ *имеем*

$$\begin{aligned} &f(t, x, y) \operatorname{sign} y \geqslant -\omega_0(|y|) \sum_{k=1}^m \psi_k(t) |y|^{\frac{1}{q}} \text{ при } a \leqslant t < \beta; \\ &f(t, x, y) \operatorname{sign} y \leqslant \omega_0(|y|) \sum_{k=1}^m \psi_k(t) |y|^{\frac{1}{q_k}} \text{ при } \alpha < t \leqslant b, \end{aligned} \tag{2.51}$$

где $a \leqslant \alpha < \beta \leqslant b$, *а* $\omega_0(t)$, $\psi_k(t)$ *и* q_k, $(k = 1,2,\ldots, m)$ *удовлетворяют условиям* (2.49), *то уравнение*

$$u'' = f(t, u, u') \tag{2.52}$$

имеет, по крайней мере, одно решение $u(t)$ *удовлетворяющее краевым условиям*

$$u(a) = l_1 u'(a) + c_1, \quad u(b) = l_2 u'(b) + c_2.$$

3. *Достаточные условия разрешимости задачи* (1.1)—(1.3)

Согласно [1, лемме 1.3], из теоремы 2.3 и ее следствии можно вывести ряд достаточных условий разрешимости задачи (1.1)—(1.3). Ниже мы приведем лишь одно из них. При этом везде будем считать, что $-\infty < a < b \leqslant +\infty$, $f_k(t, x, y) \in K_t(a, t_0)$ при любом $t_0 \in (a, b)$, $(k = 1,2)$ а функция $\varphi(t)$ непрерывна в промежутке $(-\infty, +\infty)$.

Теорема 3.1 *Если* $-\infty < \sigma_1 < \sigma_2 < +\infty$;

$$(-1)^k \varphi(x) \geqslant 0 \text{ при } (-1)^k (x - \sigma_k) > 0, \quad (k = 1, 2);$$

$$(-1)^k f_2(t, \sigma_k, 0) \geqslant 0 \text{ при } a \leqslant t < b, \ (k = 1, 2),$$

а в области $\sigma_1 \leqslant x \leqslant \sigma_2$, $-\infty < y < +\infty$ *имеем*

$$f_1(t, x, y) \operatorname{sign} y \geqslant \omega(t, |y|) \text{ при } a \leqslant t < b; \tag{3.1}$$

$$\begin{aligned} &f_2(t, x, y,) \operatorname{sign} y \geqslant -\omega_0(|y|) \sum_{k=1}^{m} \psi_k(t) [\omega(t, |y|)]^{\frac{1}{q}} \text{ при } a \leqslant t \leqslant \beta; \\ &f_2(t, x, y) \operatorname{sign} y \leqslant \omega_0(|y|) \sum_{k=1}^{m} \psi_k(t) [\omega(t, |y|)]^{\frac{1}{q_k}} \text{ при } \alpha \leqslant t < b, \end{aligned} \tag{3.2}$$

где $a \leqslant \alpha < \beta < b$; $\omega(t, y)$ *определена и неотрицательна в области* $a \leqslant t < b$, $0 \leqslant y < +\infty$; $\omega(t, |y|) \in K_t(a, t_0)$ *при любом* $t_0 \in (a, b)$;

$$\omega(t, y) > 0 \text{ при } y > 0, \text{ не убывает по } y \text{ и } \lim_{y \to +\infty} \int_{\alpha}^{\beta} \omega(t, y)\, dt = +\infty; \tag{3.3}$$

$$\begin{aligned} &\psi_k(t) \geqslant 0, \ \psi_k(t) \in L^{p_k}(a, t_0) \text{ при любом } t_0 \in (a, b), \quad \frac{1}{p_k} + \frac{1}{q_k} = 1, \\ &1 \leqslant q_k \leqslant +\infty, \quad (k = 1,2,\ldots, m); \end{aligned} \tag{3.4}$$

$$\omega_0(t) > 0 \text{ непрерывна при } 0 \leqslant t < +\infty \text{ и } \int_0^{+\infty} \frac{dt}{\omega_0(t)} = +\infty,$$

то задача (1.1)—(1.3) *имеет, по крайней мере, одно решение.*

Доказательство. В силу (3.3), найдется такое число $\alpha_0 \in (\alpha, \beta)$ что

$$\lim_{y\to+\infty} \int_{\alpha}^{\alpha_0} \omega(t, y)\, dt > \sigma_2 - \sigma_1 \text{ и } \lim_{y\to+\infty} \int_{\alpha_0}^{\beta} \omega(t, y)\, dt > \sigma_2 - \sigma_1 .$$

Поэтому, в силу [1, лемм 1.2 и 1.2′], будем иметь

$$\{\omega(t, y), \omega_1(t, y)\} \in B_1(r; a, \alpha_0, \beta), \quad \{\omega(t, y), \omega_2(t, y)\} \in B_2(r, \alpha, \alpha_0, t^*) \tag{3.5}$$
$$\text{при любом } t^* \in (\beta, b),$$

где

$$\omega_1(t, y) = \omega_2(t, y) = \omega_0(y) \sum_{k=1}^{m} \psi_k(t)\, [\omega(t, y)]^{\frac{1}{q_k}} . \tag{3.6}$$

Согласно [1, определениям 1.1 и 1.2], из (3.5) ясно, что для любого $t^* \in (\beta, b)$ найдется такое положительное число $M(t^*)$, что каковы бы ни были абсолютно непрерывные на некотором отрезке $[a, \bar{t}]$, $\beta < \bar{t} \leqslant t^*$ фунцкии $v_1(t)$ и $v_2(t)$, если $\sigma_1 \leqslant v_1(t) \leqslant \sigma_2$,

$$\begin{aligned} &v_1'(t) \operatorname{sign} v_2(t) \geqslant \omega(t, |v_2(t)|) \text{ при } a \leqslant t \leqslant \bar{t} , \\ &v_2'(t) \operatorname{sign} v_2(t) \geqslant -\omega_1(t, |v_2(t)|) \text{ при } a \leqslant t \leqslant \beta, \\ &v_2'(t) \operatorname{sign} v_2(t) \leqslant \omega_2(t, |v_2(t)|) \text{ при } \alpha \leqslant t \leqslant \bar{t} , \end{aligned} \tag{3.7}$$

то будем иметь

$$|v_1(t)| + |v_2(t)| \leqslant M(t^*) \text{ при } \alpha \leqslant t \leqslant \bar{t}.$$

Предположим теперь, что $\{b_k\}$ — некоторая возрастающая последовательность, удовлетворяющая условиям

$$\beta < b_k < b, \ (k = 1, 2, \ldots) \text{ и } \lim_{k\to\infty} b_k = b.$$

Согласно следствию 1 теоремы 2.3, легко проверить, что в условиях теоремы 3.1 при любом фиксированном к системе (1.1) имеет решение $u_{1k}(t)$, $u_{2k}(t)$ удовлетворяющее условиям

$$u_{2k}(a) = \varphi[u_{1k}(a)], \ u_{1k}(b_k) = \frac{\sigma_1 + \sigma_2}{2} , \tag{3.8}$$

$$\sigma_1 \leqslant u_{1k}(t) \leqslant \sigma_2 \text{ при } a \leqslant t \leqslant b_k . \tag{3.9}$$

Из (3.1), (3.2) и (3.6) следует, что для любого $\bar{t} \in [\beta, b_k]$ функции $v_1(t) = u_{1k}(t)$ и $v_2(t) = u_{2k}(t)$ удовлетворяют нерабенствам (3. 6). Поэтому из вышесказанного ясно, что для любого $t^* \in (\beta, b)$ найдется такое положительное число $M(t^*)$, что соблюдаются неравенства [1, (1.29)]. Следователь-

но, последовательность $\{u_{1k}(t), u_{2k}(t)\}$ удовлетворяет всем условиям [1, леммы 1.3]. Поэтому, без ограничения общности, можем считать, что она сходится к некоторому определенному в промежутке (a, b) решению $u_1(t), u_2(t)$ системы (1.1). Перейдя к пределу, когда $k \to \infty$, из (3.8) и (3.9) найдем

$$u_2(a) = \varphi\,[u_1(a)], \quad \sigma_1 \leqslant u_1(t) \leqslant \sigma_2 \text{ при } a < t < b.$$

Следовательно, $u_1(t), u_2(t)$ является решением задачи (1.1)—(1.3). Теорема доказана.

Из доказанной теоремы непосредственно вытекает такое

Следствие. *Если* $-\infty < \sigma_1 \leqslant c \leqslant \sigma_2 < +\infty$, $0 < l < +\infty$ *и соблюдаются условия* (2.50), (2.51) *и* (3.4), *то дифференциальное уравнение* (2.52) *имеет, по крайней мере, одно решение* $u(t)$ *удовлетворяющее краевым условиям*

$$u(a) = lu'(a) + c, \quad \sup_{a \leqslant t \leqslant b} |u(t)| < +\infty .$$

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ASUPRA UNOR PROBLEME LA LIMITĂ NELINIARE (II)

(Rezumat)

Utilizînd definiţiile şi rezultatele din [1] se stabilesc condiţii suficiente de existenţă a Soluţiilor pentru sistemul de ecuaţii diferenţiale (1.1), satisfăcînd una din condiţiile la limită (1.2) sau (1.3).

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ASYMPTOTIC BEHAVIOUR OF A CLASS OF NONLINEAR INTEGRAL EQUATIONS

BY

A. N. V. RAO

1. Introduction

The absolute stability of autonomous feedback control systems with multiple nonlinearities and distributed parameters, has been the subject of many investigations [1], [7], [8]. Such systems lead to the study of an integral equation of the type

$$\sigma(t) = f(t) + \int_0^t G(t-u)\, \Phi(\sigma(u))\, \mathrm{d}u. \tag{1.1}$$

In this paper, we will investigate the asymptotic behaviour of solutions of the more general class of integral equations

$$\sigma(t) = f(t, \sigma(t)) + \int_0^t G(t-u)\, \Phi(\sigma(u))\, \mathrm{d}u. \tag{1.2}$$

The main result is presented in Theorem 3.1 and involves a frequency condition on the kernal, $G(t)$.

2. Notation and Preliminaries

Let R denote the real line $(-\infty, +\infty)$ and R^+ the interval $[0, \infty)$. Let R^n denote the n-dimensional Euclidian space. We define

(i) if $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, a column n-vector, $\| x \| = \sqrt{x_1^2 + \ldots + x_n^2}$;

(ii) A^T — transpose of the matrix A, $A^* =$ conjugate transpose of A;

(iii) if A is a $n \times n$ matrix, x a column n-vector,

$$\| A \| = \max_{\| x \| = 1} \| Ax \| ;$$

(iv) $I = n \times n$ unit matrix;

(v) $F \in L_1(0, \infty) \Rightarrow F : R^+ \to R^n$ and $\int_0^\infty \| F(t) \| \, dt < \infty$;

(vi) if $F \in L_1(0, \infty)$, $\widetilde{F}(i\omega) \equiv \int_0^\infty F(t) \, e^{-i\omega t} \, dt$;

(vii) if x, y are column n-vectors

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i .$$

We also make the following assumptions with respect to (1.1):

(1) $f : R^+ \times R^n \to R^n$ is differentiable;

(2) $G : R^+ \to R^{n^2}$ is a differentiable $n \times n$ matrix;

(3) $\Phi : R^n \to R^n$ is continuous.

3. The Main Theorem

Theorem 3.1. *Let the integral equation* (1.1) *satisfy the following conditions*:

(A) (i) $\| f(t, \sigma) \| \leqslant g_1(t) + k_1 \| \sigma \|$, *where* $g_1(t)$ *is bounded and* $\in L_1(0, \infty)$ *and* $0 \leqslant k_1 < 1$,

(ii) $\left\| \dfrac{\partial f}{\partial t} \right\| \leqslant g_2(t) + k_2 \| \sigma \|$, *where* $g_2(t)$ *is bounded and* $\in L_1(0, \infty)$ *and* $k_2 \in R^+$,

(iii) $\langle f(t, \sigma), \Phi(\sigma) \rangle \leqslant g_3(t) \| \Phi(\sigma) \| + k_1 \langle \sigma, \Phi(\sigma) \rangle$, *where* $g_3(t) \in L_1(0, \infty)$ *and* k_1 *is as defined in* (i) *above*,

(iv) *Each characteristic root*, $\lambda(\sigma, t)$, *of the matrix* $\left[\dfrac{\partial f}{\partial \sigma}\right]^T \left[\dfrac{\partial f}{\partial \sigma}\right]$, *satisfies the inequality* $\lambda(\sigma, t) \leqslant \gamma$, *for* $t \in R^+$, $\sigma \in R^n$, *where* $0 \leqslant \gamma < 1$;

(B) $G(t),\ G'(t) \in L_1(0, \infty) \cap L_2(0, \infty)$;

(C) $\Phi(\sigma) = \begin{bmatrix} \Phi_1(\sigma_1) \\ \vdots \\ \Phi_n(\sigma_n) \end{bmatrix}$ *satisfies the condition*

$0 < \varepsilon_{ii}\sigma_i^2 \leqslant \Phi_i(\sigma_i)\sigma_i \leqslant (k_{ii} - \varepsilon_{ii})\sigma_i^2$ *for some* $k_{ii} \in R^+$, $(i = 1, 2, ..., n)$, ε_{ii} *is a positive, arbitrarily small real number* ;

(D) $\exists$ *a constant, positive, real diagonal matrix* Q *such that*

$$\mathcal{R}e\left\{(I + i\omega Q)\widetilde{G}(i\omega) + \frac{kk_2\|Q\|}{(1-k_1)^2}\widetilde{G}(i\omega)\widetilde{G}^*(i\omega)\right\} + (1-k_1)K^{-1} \leqslant 0,$$

where $K = \begin{bmatrix} k_{11} & & 0 \\ & \ddots & \\ 0 & & k_{nn} \end{bmatrix}$, $k_{ii} > 0$ *and* $k = \max\limits_{(i=1,2,...,n)} k_{ii}$.

Then every solution, $\sigma(t)$, of (1.1) *satisfies the condition* $\lim\limits_{t\to\infty} |\sigma(t)| = 0$.

Remark. Although we have not proved the existence of solutions for (1.1), it can be shown [6] that the hypothesis of theorem and the assumptions that we had made earlier with respect to (1.1), assure the global existence of solutions for (1.1).

P r o o f. Define

$$\Phi_t(\tau) = \begin{cases} \Phi(\sigma(\tau)), & 0 \leqslant \tau \leqslant t, \\ 0, & \tau > t; \end{cases}$$

$$f_t(\tau) = \begin{cases} f(\tau, \sigma(\tau)), & 0 \leqslant \tau \leqslant t, \\ 0, & \tau > t; \end{cases}$$

$$\frac{\partial f_t}{\partial \tau} = \begin{cases} \dfrac{\partial f}{\partial \tau}(\tau, \sigma), & 0 \leqslant \tau \leqslant t, \\ 0, & \tau > t; \end{cases}$$

$$\frac{\partial f_t}{\partial \sigma} = \begin{cases} \dfrac{\partial f}{\partial \sigma}(\tau, \sigma) & 0 \leqslant \tau \leqslant t, \\ 0 & \tau > t; \end{cases}$$

$$\sigma_t(\tau) = f_t(\tau) + \int_0^\tau G(\tau - u)\,\Phi_t(u)\,du\,;$$

$$\sigma_t'(\tau) = \frac{\partial f_t}{\partial \tau} + \frac{\partial f_t}{\partial \sigma}\frac{d\sigma}{d\tau} + G(0)\,\Phi_t(\tau) + \int_0^\tau G'(\tau - u)\Phi_t(u)\,du.$$

Let

$$\Gamma_t(\tau) = \int_0^\tau \left\{ G(\tau - u) + QG'(\tau - u) \right\} \Phi_t(u) \mathrm{d}u + \tag{3.1}$$
$$+ QG(0)\Phi_t(\tau) - (1 - k_1)K^{-1}\Phi_t(\tau).$$

Using the definitions of the different truncated functions, it is easily verified that

$$\Gamma_t(\tau) = \sigma_t(\tau) + Q\sigma_t'(\tau) - f_t(\tau) - Q\left(\frac{\partial f_t}{\partial \tau} + \frac{\partial f_t}{\partial \sigma}\frac{\mathrm{d}\sigma}{d\tau}\right) - (1-k_1)K^{-1}\Phi_t(\tau). \tag{3.2}$$

Put

$$\alpha(t) = \int_0^t \left\langle \Gamma_t(\tau), \Phi_t(\tau) \right\rangle \mathrm{d}\tau + kk_2 \| Q \| \int_0^t \left\langle \sigma_t(\tau), \sigma_t(\tau) \right\rangle \mathrm{d}\tau - \tag{3.3}$$
$$- \frac{2m_1k^2}{(1-k_1)^2} k_2 \| Q \| \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| - \frac{m_2 k}{(1-k_1)^2} k_2 \| Q \|,$$

where

$$m_1 = \int_0^\infty \{g_1(t)\} \, \mathrm{d}t \int_0^\infty \| G(\tau) \| \, \mathrm{d}\tau \tag{3.4}$$

and

$$m_2 = \int_0^\infty \{g_1(\tau)\}^2 \, \mathrm{d}\tau. \tag{3.5}$$

(Note by assumption that $g_1(t)$ is bounded and $\in L_1(0, \infty)$. This implies that $g_1(t) \in L_2(0, \infty)$).

It is shown in Appendix I that

$$kk_2 \| Q \| \int_0^t \left\langle \sigma_t(\tau), \sigma_t(\tau) \right\rangle \mathrm{d}\tau \leqslant \frac{m_2 k k_2}{(1-k_1)^2} \| Q \| +$$
$$+ \left\{ \frac{2m_1 k^2}{(1-k_1)^2} k_2 \| Q \| \right\} \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| + \tag{3.6}$$
$$+ \frac{kk_2 \| Q \|}{(1-k_1)^2} \int_0^t \left\langle \int_0^\tau G(\tau - u) \, \Phi_t(u) \, \mathrm{d}u, \int_0^\tau G(\tau - u) \, \Phi_t(u) \, \mathrm{d}u \right\rangle \mathrm{d}\tau.$$

Using (3.6), we can write (3.3) as

$$\alpha(t) \leqslant \int_0^t \langle \Gamma_t(\tau), \Phi_t(\tau) \rangle \, d\tau + \tag{3.7}$$
$$+ \frac{kk_2 \| Q \|}{(1-k_1)^2} \int_0^t d\tau \left\langle \int_0^\tau G(\tau - u)\Phi_t(u)\, du, \int_0^\tau G(\tau - u)\Phi_t(u)\, du \right\rangle .$$

Substituting in (3.7) for $\Gamma_t(\tau)$ from (3.1), one has

$$\alpha(t) \leqslant \int_0^t \left\langle \int_0^\tau \{G(\tau - u) + QG'(\tau - u)\}\Phi_t(u)\, du, \Phi_t(\tau) \right\rangle d\tau +$$
$$+ \int_0^t \langle QG(0)\Phi_t(\tau), \Phi_t(\tau) \rangle \, d\tau + \frac{kk_2 \| Q \|}{(1-k_1)^2} \int_0^t d\tau \left\langle \int_0^\tau G(\tau - u)\Phi_t(u)\, du, \right. \tag{3.8}$$
$$\left. \int_0^\tau G(\tau - u)\Phi_t(u) du \right\rangle - \int_0^t d\tau \langle (1 - k_1)K^{-1}\Phi_t(\tau), \Phi_t(\tau) \rangle .$$

It is easy to see that the right hand side of (3.8) $\in L_1(0, \infty) \cap L_2(0, \infty)$. Noting that all the functions involved are real, we get, on applying P a r s e v a l's Theorem to the right hand side of (3.8),

$$\alpha(t) \leqslant \int_{-\infty}^{+\infty} \mathcal{R}e \left\langle \left[\left\{ (I + j\omega Q)\tilde{G}(j\omega) + \frac{kk_2 \| Q \|}{(1-k_1)^2} \tilde{G}(j\omega)\tilde{G}^*(j\omega) \right\} \tilde{\Phi}(j\omega) - \right. \right. \tag{3.9}$$
$$\left. \left. - (1 - k_1)K^{-1}\tilde{\Phi}(j\omega) \right], \tilde{\Phi}(-j\omega) \right\rangle d\omega.$$

By assumption (E_1), the integral on the right hand side is nonpositive. So we conclude

$$\alpha(t) \leqslant 0. \tag{3.10}$$

On the other hand it is shown in Appendix II that

$$\alpha(t) \geqslant (1 - k_1) \int_0^t \langle \sigma(\tau) - K^{-1}\Phi(\sigma(\tau)), \Phi(\sigma(\tau)) \rangle \, d\tau + \tag{3.11}$$
$$+ a \| \sigma(t) \|^2 - b \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| - c,$$

where the constants a, b and c are defined respectively by (II.6), (II.7) and (II.8) in Appendix II. The first integral of (3.11) is, by assumption (C), positive and hence can be dropped from (3.11). Thus we get

$$a\|\sigma(t)\|^2 - b \sup_{0 \leqslant \tau \leqslant t} \|\sigma(\tau)\| - c \leqslant 0. \tag{3.12}$$

Since $\|\sigma(t)\|$ is continuous, $\exists\, a\ T = T(t) \in [0, t]$ where $\|\sigma(t)\|$ attains its supremum. As the inequality (3.12) is true for every $t \in [0, t]$, in particular it is true for $t = T$ and so we have

$$a\|\sigma(T)\|^2 - b\|\sigma(T)\| - c \leqslant 0, \tag{3.13}$$

which implies

$$\|\sigma(t)\| \leqslant \|\sigma(T)\| \leqslant \frac{b + \sqrt{b^2 + 4ac}}{2a}. \tag{3.14}$$

It follows that $\|\sigma(t)\|$ is uniformly bounded and consequently $\|\Phi(\sigma(t)\|$ is uniformly bounded. From (1.1) one has

$$(1 - \sqrt{\gamma})\|\sigma'(t)\| \leqslant g_2(t) + k_2\|\sigma(t)\| + \sup_{t \geqslant 0} \|\Phi(\sigma(t))\| \int_0^\infty \|G(u)\| \mathrm{d}u.$$

We conclude that $\|\sigma'(t)\|$ is uniformly bounded as $(1 - \sqrt{\gamma})$ is positive and every term on the right hand side is bounded. This implies that $\sigma(t)$ is uniformly continuous.

Now consider the inequality (3.11). As $\|\sigma(t)\|$ is uniformly bounded it follows that $(1 - k_1) \int_0^t \langle \sigma(\tau) - K^{-1}\Phi(\sigma(\tau)), \Phi(\sigma(\tau)) \rangle \mathrm{d}\tau$ is uniformly bounded. Using the assumption (C), we conclude that

$$\frac{\varepsilon_{i\,i_{\min}}^2}{k_{i\,i_{\max}}} \int_0^t \|\sigma(\tau)\|^2 \mathrm{d}\tau$$

is bounded. As $\sigma(t)$ is uniformly continuous it follows from B a r b a l a t's Lemma [2]

$$\lim_{t \to \infty} \|\sigma(t)\| = 0.$$

Remark. (i) If $f(t, \sigma) = \begin{bmatrix} f_1(t, \sigma_1) \\ \vdots \\ f_n(t, \sigma_n) \end{bmatrix}$, it is easily shown that assumption (A) (i) implies (A) (iii).

(ii) If σ is a scalar, assumption (A) (iii) can be slightly weakened to $\frac{\partial f}{\partial t} \leqslant g_2(t) + k_2 \| \sigma \|$. In this case it is also easy to give a geometric interpretation of the frequency condition similar to that given in [3].

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Appendix I

It follows from the definitions of $\sigma_t(\tau)$ that

$$\int_0^t \langle \sigma(\tau), \sigma(\tau) \rangle \mathrm{d}\tau = \int_0^t \langle \sigma_t(\tau), \sigma_t(\tau) \rangle \mathrm{d}\tau = \int_0^\infty \langle \sigma_t(\tau), \sigma_t(\tau) \rangle \mathrm{d}\tau.$$

From (1.1), we obtain on using the assumption (A) (i)

$$(1 - k_1) \| \sigma(\tau) \| \leqslant g_1(\tau) + \left\| \int_0^\tau G(\tau - u) \Phi(\xi(u)) \mathrm{d}u \right\|.$$

Thus

$$kk_2 \| Q \| \int_0^t \langle \sigma_t(\tau), \sigma_t(\tau) \rangle \mathrm{d}\tau = kk_2 \| Q \| \int_0^t \langle \sigma(\tau), \sigma(\tau) \rangle \mathrm{d}\tau \leqslant$$

$$\leqslant \frac{kk_2 \| Q \|}{(1-k_1)^2} \int_0^\infty \{ g_1(\tau) \}^2 \mathrm{d}\tau + \frac{2k^2 k_2 \| Q \|}{(1-k_1)^2} \int_0^t g_1(\tau) \mathrm{d}\tau \int_0^\tau \| G(\tau - u) \| \, \| \sigma(u) \| \mathrm{d}u +$$

$$+ \frac{kk_2 \| Q \|}{(1-k_1)^2} \int_0^t \mathrm{d}\tau \left\langle \int_0^\tau G(\tau - u) \Phi(\sigma(u)) \mathrm{d}u, \int_0^\tau G(\tau - u) \Phi(\sigma(u)) \mathrm{d}u \right\rangle \leqslant$$

$$\leqslant \frac{kk_2 \| Q \|}{(1-k_1)^2} \int_0^\infty \{ g_1(\tau) \}^2 \mathrm{d}\tau + \frac{2k^2 k_2 \| Q \|}{(1-k_1)^2} \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| \int_0^\infty \{ g_1(\tau) \} \mathrm{d}\tau \int_0^\infty \| G(\tau) \| \mathrm{d}\tau \times$$

$$\left\{ \int_0^t \mathrm{d}\tau \left\langle \int_0^\tau G(\tau - u) \Phi(\sigma(u)) \mathrm{d}u, \int_0^\tau G(\tau - u) \Phi(\sigma(u)) \mathrm{d}u \right\rangle \right\} \frac{kk_2 \| Q \|}{(1-k_1)^2} =$$

$$= \frac{m_2 kk_2 \| Q \|}{(1-k_1)^2} + \frac{2m_1 k^2 k_2 \| Q \|}{(1-k_1)^2} \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| + \frac{kk_2 \| Q \|}{(1-k_1)^2} \times$$

$$\times \left[\int_0^t \mathrm{d}\tau \left\langle \int_0^\tau G(\tau - u)\Phi(\sigma(u))\mathrm{d}u, \int_0^\tau G(\tau - u)\Phi(\sigma(u))\mathrm{d}u \right\rangle \right],$$

where we have put for notational convenience

$$\int_0^\infty \{g_1(\tau)\}^2 \, \mathrm{d}\tau \int_0^\infty \| G(\tau) \| \, \mathrm{d}\tau = m_1$$

and

$$\int_0^\infty \{g_1(\tau)\}^2 \mathrm{d}\tau = m_2.$$

Appendix II

$$\text{(II.1)} \quad \begin{aligned} \alpha(t) = & \int_0^t \left\langle \Gamma_t(\tau), \Phi_t(\tau) \right\rangle \mathrm{d}\tau + kk_2 \|Q\| \int_0^t \left\langle \sigma_t(\tau), \sigma_t(\tau) \right\rangle \mathrm{d}\tau - \\ & - \frac{2m_1k^2}{(1-k_1)^2} k_2 \|Q\| \sup_{0 \leqslant \sigma \leqslant t} \| \sigma(\tau) \| - \frac{m_2k}{(1-k_1)^2} k_2 \|Q\|. \end{aligned}$$

Using (3.2), (II.1) can be written, for $0 \leqslant \tau \leqslant t$, as

$$\text{(II.2)} \quad \begin{aligned} \alpha(t) = & \int_0^t \left\langle \sigma(\tau) - (1-k_1)K^{-1}\Phi(\sigma(\tau)) - f(\tau, \sigma(\tau)), \Phi(\sigma(\tau)) \right\rangle \mathrm{d}\tau + \\ & + \int_0^t \left\langle Q\sigma'(\tau), \Phi(\sigma(\tau)) \right\rangle \mathrm{d}\tau - \int_0^t \left\langle Q \frac{\partial f}{\partial \tau} + Q \frac{\partial f}{\partial \sigma} \frac{\mathrm{d}\sigma}{\mathrm{d}t}, \Phi(\sigma(\tau)) \right\rangle \mathrm{d}\tau + \\ & + kk_2 \| Q \| \int_0^t \left\langle \sigma(\tau), \sigma(\tau) \right\rangle \mathrm{d}\tau - \frac{2m_2k}{(1-k_1)^2} k_2 \| Q \| \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| - \\ & - \frac{m_2k}{(1-k_1)^2} k_2 \| Q \|. \end{aligned}$$

Using the assumptions (A) (i), (A) (ii) and (A) (iii), we get

$$\int_0^t \left\langle Q\frac{\partial f}{\partial \tau}, \Phi(\sigma(\tau))\right\rangle d\tau \leqslant k\| Q\| \sup_{0\leqslant\tau\leqslant t}\|\sigma(\tau)\| \int_0^\infty g_2(u)du + \tag{II.3}$$

$$+ kk_2\| Q\| \int_0^t \left\langle \sigma(\tau), \sigma(\tau)\right\rangle d\tau$$

and

$$\alpha(t) \geqslant \int_0^t \left\langle \sigma(\tau) - k_1\sigma(\tau) - K^{-1}(1-k_1)\Phi(\sigma(\tau)), \Phi(\sigma(\tau))\right\rangle d\tau -$$

$$- k \sup_{0\leqslant\sigma\leqslant t}\|\sigma(\tau)\| \int_0^\infty g_3(\tau)\, d\tau + \int_0^t \left\langle Q\sigma'(\tau), \Phi(\sigma(\tau))\right\rangle d\tau - \tag{II.4}$$

$$- \int_0^t \left\langle Q\frac{\partial f}{\partial \sigma}\frac{d\sigma}{d\tau}, \Phi(\sigma(\tau))\right\rangle d\tau - k\| Q\| \sup_{0\leqslant\tau\leqslant t}\|\sigma(\tau)\| \int_0^\infty g_2(\tau)\, d\tau -$$

$$- \frac{2m_1k^2}{(1-k_1)^2} k_2\| Q\| \sup_{0\leqslant\tau\leqslant t}\|\sigma(\tau)\| - \frac{m_2k}{(1-k_1)^2} k_2\| Q\|,$$

$$\alpha(t) \geqslant (1-k_1)\int_0^t \left\langle \sigma(\tau) - K^{-1}\Phi(\sigma(\tau)), \Phi(\sigma(\tau))\right\rangle d\tau - k + g_3(\tau) +$$

$$+ \| Q\| g_2(\tau) \sup_{0\leqslant\tau\leqslant t}\|\sigma(\tau)\| (\sigma_1(t), \ldots, \sigma_m(t)) +$$

$$+ \int_{(\sigma_1(0),\ldots,\sigma_m(0))}^{(\sigma_1(t),\ldots,\sigma_m(t))} \left\langle Q\left(I - \frac{\partial f}{\partial u}\right) du, \Phi(u)\right\rangle - \frac{2m_1k^2}{(1-k_1)^2} k_2\| Q\| \sup_{0\leqslant\tau\leqslant t}\|\sigma(\tau)\| (\sigma_1(0), \ldots, \sigma_m(0)) -$$

$$- \frac{m_2k}{(1-k_1)^2} k_2\| Q\|.$$

By assumption (A) (IV), it follows that for each characteristic root, $\lambda_1(u, t)$, of $\left\|\frac{\partial f}{\partial u}\right\|$, the inequality $|\lambda_1(u, t)| \leqslant \sqrt{\gamma}$ holds. Consequently each characteristic root, $\lambda_2(u, t)$, of the matrix $\left[I - \frac{\partial f}{\partial u}\right]$ satisfies the condition

$$|\lambda_2(u, t)| \geqslant 1 - \sqrt{\gamma} > 0.$$

This, together with the facts that Q is positive definite and

$$\int_0^{\sigma} \left\langle \Phi(u), \mathrm{d}u \right\rangle > 0,$$

implies

$$\text{(II.5)} \quad \int_{(\sigma_1(0),\ldots,\sigma_m(0))}^{(\sigma_1(t),\ldots,\sigma_m(t))} \left\langle Q\left(\mathrm{I} - \frac{\partial f}{\partial u}\right) \mathrm{d}u, \Phi(u) \right\rangle \geqslant (1 - \sqrt{\gamma}) q_{ii\min} \int_{(\sigma_1(0),\ldots,\sigma_m(0))}^{1(t),\ldots,\sigma_m(t))} \left\langle \Phi(u), \mathrm{d}u \right\rangle \geqslant$$

$$\geqslant (1 - \sqrt{\gamma})\, q_{ii\min} \frac{\varepsilon_{ii\min}}{2} \| \sigma(t) \|^2 - (1 - \sqrt{\gamma}) q_{ii\min} \frac{k}{2} \| \sigma(0) \|^2,$$

because

$$\int_{(\sigma_1(0),\ldots,\sigma_m(0))}^{(\sigma_1(t),\ldots,\sigma_m(t))} \left\langle \Phi(u), \mathrm{d}u \right\rangle = \int_{(\sigma_1(0),\ldots,\sigma_m(0))}^{(0,0,\ldots,0)} \left\langle \Phi(u), \mathrm{d}u \right\rangle + \int_{(0,0,\ldots,0)}^{(\sigma_1(t)\ldots,\sigma_m(t))} \left\langle \Phi(u), \mathrm{d}u \right\rangle$$

and by assumption

$$\int_0^{\sigma_i(t)} \Phi_i(u_i) \mathrm{d}u_i \geqslant \frac{\varepsilon_{ii}}{2} (\sigma_i(t))^2 \quad \text{and} \quad \int_0^{\sigma_i(0)} \Phi_i(u_i) \mathrm{d}u_i < \frac{\varepsilon_{ii}}{2} (\sigma_i(0))^2.$$

Using (II.5) in (II.4), we get

$$\alpha(t) \geqslant (1 - k_1) \int_0^t \left\langle \sigma(\tau) - K^{-1}\Phi(\sigma(\tau)), \Phi(\sigma(\tau)) \right\rangle \mathrm{d}\tau - k + g_3(\tau) +$$

$$+ \| Q \| \int_0^t g_2(\tau)\, \mathrm{d}\tau + \frac{2 m_1 k^2 k_2}{(1 - k_1)^2} \| Q \| \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| +$$

$$+ \frac{(1 - \sqrt{\gamma})}{2} q_{ii\min}\, \varepsilon_{ii\min} \| \sigma(t) \|^2 - \frac{(1 - \sqrt{\gamma})}{2} q_{ii\min}\, k \| \sigma(0) \|^2 -$$

$$- \frac{m_2 k k_2}{(1 - k_1)^2} \| Q \| = (1 - k_1) \int_0^t \left\langle \sigma(\tau) - K^{-1}\Phi(\sigma(\tau)), \Phi(\sigma(\tau)) \right\rangle \mathrm{d}\tau +$$

$$+ a \| \sigma(t) \|^2 - b \sup_{0 \leqslant \tau \leqslant t} \| \sigma(\tau) \| - c,$$

where, for notational convenience we have defined

$$a = \frac{(1 - \sqrt{\gamma})}{2} q_{ii\min} \varepsilon_{ii\min}, \tag{II.6}$$

$$b = k \left\{ \int_0^\infty [g_3(\tau) + \| Q \| g_2(\tau)] \, d\tau + \frac{2m_1 k^2 k_2}{(1 - k_1)^2} \| Q \| \right\}, \tag{II.7}$$

$$c = \frac{(1 - \sqrt{\gamma})}{2} q_{ii\min} k \| \sigma(0) \|^2 + \frac{m_2 k k_2}{(1 - k_1)^2} \| Q \|. \tag{II.8}$$

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COMPORTAREA ASIMPTOTICĂ A UNEI CLASE DE ECUAŢII INTEGRALE NELINIARE

(Rezumat)

Se dau condiţii suficiente pentru ca soluţiile ecuaţiei integrale (1.2) să tindă asimptotic spre zero, pentru o clasă de funcţii Φ. Astfel de ecuaţii intervin în teoria reglării automate. Condiţiile care se obţin implică o condiţie de tip frecvenţial pentru nucleul G, care permite o interpretare geometrică atunci cînd $\tau(t)$ este funcţie scalară.

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ПРИБЛИЖЕННОЕ РЕШЕНИЕ НЕЛИНЕЙНЫХ СИСТЕМ ВТОРОГО ПОРЯДКА С НАСЛЕДСТВЕННОСТЬЮ ОДНОГО НЕЗАВИСИМОГО ПЕРЕМЕННОГО

Л. Е. КРИВОШЕИН, Д. МАНЖЕРОН и М. Н. ОГЮЗТОРЕЛИ

Академику И. Н. Векуа к его юбилею

1. В ряде работ авторов посвященных исследованию математических систем сложных структур [1]... [4] были изучены линейные и нелинейные системы с наследственными операторами, с запаздывающим аргументом, одного и многих переменных, частично использованные затем другими исследователями [5] — [6].

В настоящей работе рассматривается вопрос нахождения приближенных решений систем второго порядка с наследственностью одного независимого переменного используя вариант метода М. Пиконе [7] или же метод верхних и нижних функций С. А. Чаплыгина [8].

Рассмотрим интегро-дифференциальное уравнение

$$(1)\qquad L[y] \equiv y''(x) + \sum_{0}^{1} R_i[x, y(x), y'(x)]y^{(i)}(x) - \int_a^x K[x, t, y(t), y'(t)]\,dt = 0,$$

при дополнительных условиях

$$(2)\qquad y^{(i)}(a) = y_0^{(i)}, \qquad (i = 0, 1).$$

Функции $R_i(\cdot)$,$(i = 0, 1)$, $K(\cdot)$ — определены и непрерывны в области D. Приближенное решение задачи (1)—(2) будем строить, пользуясь идеей М. Пиконе [7] о представлении искомой функции в виде суммы известной функции и новой неизвестной функции с таким расчетом, чтобы основная часть искомой функции приходилась на известную функцию. Предполагается при этом, что задача (1), (2) имеет единственное решение в классе $C_2[a, b]$. С этой целью разобъем сегмент $[a, b]$ где ищется

$y(x)$, на части точками $a = x_0 < x_1 < \ldots < x_n = b$ и вместо задачи (1)—(2) будем рассматривать следующую систему задач:

$$y_1''(x) + \sum_0^1 R_i[a,\ y_0,\ y_0']\, y_1^{(i)}(x) = K[x,\ a,\ y_0,\ y_0'](x - a), \tag{3}$$

$$y_1^{(i)}(a) = y_0^{(i)}, \qquad (i = 0,\ 1), \tag{4}$$

если $a \leqslant x \leqslant x_1$; и

$$\begin{gathered} y_k''(x) + \sum_0^1 R_i[x_{k-1},\ y_{k-1}(x_{k-1}),\ y_{k-1}'(x_{k-1})]\, y^{(i)}(x) = \\ = K[x,\ x_{k-1},\ y_{k-1}(x_{k-1}),\ y_{k-1}'(x_{k-1})](x - x_{k-1})^{1)} + \\ + \sum_{i=1}^{k-1} \int_{x_{i-1}}^{x_i} K[x,\ t,\ y_i(t),\ y_i'(t]\ \mathrm{d}t \equiv F_k(x), \end{gathered} \tag{5}$$

$$y_{k-1}^{(i)}(x_{k-1}) = y_k^{(i)}(x_{k-1}), \qquad (i = 0,\ 1), \tag{6}$$

если $x_{k-1} \leqslant x \leqslant x_k$, $(k \geqslant 2)$.

Из (3), (4) и (5), (6) легко находим $y_1(x) = P_1(x)$, $y_k(x) = P_k(x)$. Функцию

$$Y_n(x) = \{P_i(x),\quad x_{i-1} \leqslant x \leqslant x_i,\quad i = \overline{1,\ n}\} \tag{7}$$

принимаем за приближенное решение исходной задачи (1), (2).

2. Функции $P_i(x)$ близкие к решению задачи (1), (2) на отдельных участках $[x_{i-1},\ x_i]$, можно строить и следующим образом, выбранным из ансамбля других возможных путей. В частности можно определять их из уравнений (4) и

$$y_1''(x) + \sum_0^1 R_i[x,\ y_0,\ y_0']\, y_1^{(i)}(x) = (x - a)\, K(x,\ a,\ y_0,\ y_0') \tag{8}$$

и (6) и

$$y_k''(x) + \sum_0^1 R_i[x, y_{k-1}(x_{k-1}),\ y_{k-1}'(x_{k-1})]\ y_{10}^{(i)}(x) = F_k(x), \tag{9}$$

при условии, что можно найти фундаментальные системы решений последних уравнений.

3. Переходим теперь к вопросу о сходимости функции (7) к точному решению задачи (1), (2). Из (3), (4) имеем

---

1) $F_1(x)$.

$$y_1(x) - y(x) = \int_a^x (x-t)\Big\{\sum_0^1 R_i[a,\ y_0,\ y_0']\, y_1^{(i)}(t) - R_i[t,\ y(t),\ y'(t)]\, y^{(i)}(t) +$$

(10)

$$+\int_0^t \{K[t,\ a,\ y_0, y_0'] - K[t,\ \tau,\ y(\tau),\ y'(\tau)]\}\, \mathrm{d}\tau\Big\}\, \mathrm{d}t$$

и таким образом

(11) $$|\, y_1(x) - y(x)\,| \leqslant O(h),$$

где $h = \max\,(x_i - x_{i-1})_1^n$, $a \leqslant x \leqslant x_1$. Если $k \geqslant 2$, то

$$y_k(x) - y(x) = \sum_0^1 (x - x_{k-1})^i [y_{k-1}(x_{k-1}) - y(x_{k-1})]^{(i)} +$$

$$+\int_{x_{k-1}}^x (x-t)\Big\{\sum_0^1 R_i[x_{k-1}, y_{k-1}(x_{k-1}), y_{k-1}'(x_{k-1})] P_k(t) - R_i[t, y(t), y'(t)]\, y^{(i)}(t) +$$

(12)

$$+\sum_{i=1}^{k-1}\int_{x_{i-1}}^{x_i} \{K[t,\ \tau,\ y_i(\tau),\ y_i'(\tau)] - K[t,\ \tau,\ y(\tau),\ y'(\tau)]\}\, \mathrm{d}\tau +$$

$$+\int_{x_{i-1}}^{x_i} \{K[t, x, y_{k-1}(x_{k-1}), y_{k-1}'(x_{k-1})] - K[t, \tau, y_i(\tau), y'(\tau)]\}\, \mathrm{d}\tau\Big\}\, \mathrm{d}t,\ \ x_{k-1} \leqslant x \leqslant x_k$$

Отсюда следует, что

(13) $$|\, y_k(x) - y(x)\,| \leqslant O(h), \qquad x_{k-1} \leqslant x \leqslant x_k.$$

Из оценок (13) и (11) видна сходимость приближенного решения (7) к точному решению исходной задачи (1), (2), когда длины частичных подсегментов $[x_i,\ x_{i+1}]$ неограниченно уменьшаются, $\lim\limits_{h\to 0} Y_n(x) = y(x)$.

4. Если функции $R_i(\cdot)$, $(i=0,\ 1)$ и $K(\cdot)$ удовлетворяют прежним условиям в области D по y и y' и эти производные неотрицательны, т.е. $\partial R_i/\partial y^{(i)} \geqslant 0$, $\partial K/\partial y^{(i)} \geqslant 0$, $(i = 0,\ 1)$, то решение задачи (1), (2) нетрудно аппроксимировать с помощью верхних и нижних функций С. А. Чаплыгина [8]. Действительно, пусть функции $u_0(x)$, $v_0(x)$ дважды непрерывно дифференцируемы на $[a,\ b]$, удовлетворяют начальным условиям (2) и следующим неравенствам

(14) $$L[u_0] = \alpha_0(x) \leqslant 0, \qquad L[v_0] = \beta_0(x) \geqslant 0.$$

Тогда из (14), пользуясь прежними результатами авторов, следует, что $u_0(x) \leqslant y(x) \leqslant v_0(x)$, $x \in [a, b]$. Нижние и верхние функции, $u_k(x)$ и $v_k(x)$, будем сближать с $y(x)$, пользуясь формулами

$$(15)\qquad \begin{cases} u_k(x) = \sum\limits_0^1 y_0^{(i)}(x-a) + \int\limits_a^x (x-t) S[u_{k-1}]\,\mathrm{d}t, \\ v_k(x) = \sum\limits_0^1 y_0^{(i)}(x-a) + \int\limits_a^x (x-t) S[v_{k-1}]\mathrm{d}t, \end{cases}$$

где

$$(16)\qquad S[\omega] \equiv \int\limits_a^x K[t, \tau, \omega(\tau), \omega'(\tau)]\,\mathrm{d}\tau - \sum_0^1 R_i[t, \omega(t), \omega'(t)]\,\omega^{(i)}(t).$$

Из (15) и (16) видно, что последовательности функций $\{u_k(x)\}$ и $\{v_k(x)$ сходятся к решению $y(x)$ задачи (1), (2)

$$(17)\qquad \lim_{k\to\infty} u_k(x) = \lim_{k\to\infty} v_k(x) = y(x),$$

а допускаемую на k-ом шаге погрешности можно оценить, исходя и выражения

$$(18)\qquad \left.\begin{matrix} |y(x) - u_k(x)| \\ |y(x) - v_k(x)| \end{matrix}\right\} \leqslant |u_k(x) - v_k(x)|, \qquad a \leqslant x \leqslant b.$$

Замечание. Известно, что в прямом методе Пиконе интегро[3] дифференциальный оператор должен быть разбит на две части с таким расчетом, чтобы одну часть можно было решить в замкнутом виде, а другую часть — приближенно с любой степенью точности. В рассмотренном нами варианте вторая часть оператора считается равной нулю и т.о., аппроксимация решения задачи (1), (2) выполняется только за счет решений линейных уравнений (3) и (5).

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SOLUŢII APROXIMATIVE ALE SISTEMELOR NELINIARE DE ORDINUL AL DOILEA CU OPERATORI EREDITARI CU O SINGURĂ VARIABILĂ INDEPENDENTĂ

(Rezumat)

Într-un şir de lucrări anterioare [1]...[4] autorii au studiat sisteme matematice cu structură complexă, folosite apoi şi de alţi cercetători [5]...[6]. În această lucrare se studiază problema determinării soluţiilor aproximative ale sistemelor de ordinul al doilea cu operatori ereditari cu o singură variabilă independentă folosindu-se o variantă a metodei lui M. Picone [7] precum şi o metodă a lui S. A. Ceaplîghin [8].

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D. K. 517.946

ZUR THEORIE DER EXPONENTIALOPERATOREN UND DER BERGMAN-OPERATOREN 1. ART

VON

ERWIN KREYSZIG

1. Einleitung

Bergman-Operatoren sind lineare Integraloperatoren vom Raum der komplex-analytischen Funktionen einer oder mehrerer Veränderlicher in den Raum der Lösungen einer gegebenen linearen homogenen partiellen Differentialgleichung. Mit ihrer Hilfe gewinnt man, indem man Methoden und Ergebnisse der Funktionentheorie benutzt, Aussagen über Klassen von Lösungen der genannten Gleichung, z. B. bezüglich der Art und Lage der Singularitäten, des Wachstums usw. Dabei handelt es sich um die Kennzeichnung allgemeiner Eigenschaften ohne Bezugnahme auf Randbedingungen.

Einen Überblick über die wichtigsten Bergman-Operatoren, die bisher definiert wurden, gibt Bergmans zusammenfassender Bericht [1]. Vgl. auch Bergman [2], [3], [4]. Wie man daraus ersieht, wurden diese Operatoren dem jeweiligen Zweck angepasst, also einzeln und unabhängig voneinander eingeführt.

Unter den Bergman-Operatoren für Differentialgleichungen in *zwei* unabhängigen Veränderlichen sind die Operatoren 1. Art und die Exponentialoperatoren (Definitionen siehe unten) am bedeutsamsten und bisher am meisten untersucht worden. Diese beiden Typen von Operatoren entsprangen ganz verschiedenartigen Problemstellungen. In der vorliegenden Arbeit wird gezeigt, dass zwischen den beiden Operatorenarten in vielen Fällen bemerkenswerte Zusammenhänge bestehen.

Gleichzeitig ergibt sich dabei für die Exponentialoperatoren eine genauere Kennzeichnung der Klasse partieller Differentialgleichungen, deren Lösungen sich vermöge solcher Operatoren darstellen lassen, und für die Operatoren 1. Art eine Möglichkeit, deren Erzeugende, die man zunächst als unendliche Reihe erhält, bei gewissen Gleichungen in geschlossener Form darzustellen.

2. Bergman-Operatoren 1. Art

Wir betrachten partielle Differentialgleichungen der Form

$$\Delta\psi + \alpha(x, y)\psi_x + \beta(x, y)\psi_y + \gamma(x, y)\psi = 0$$

mit analytischen Koeffizienten. Sind x und y komplex, so sind $z = x + iy$ und $z^* = x - iy$ unabhängige Variable. Für z und z^* gewinnt die obige Gleichung nach Elimination einer der beiden ersten partiellen Ableitungen die Form

$$Lu = u_{zz^*} + b(z, z^*)u_{z^*} + c(z, z^*)u = 0\,; \tag{2.1}$$

$b(z, z^*)$ und $c(z, z^*)$ sind dabei analytisch.

Den *Trivialfall* $c \equiv 0$, in dem sich (2.1) auf eine gewöhnliche Differentialgleichung für u_{z^*} reduziert, schliessen wir stets aus.

Für die Gleichung (2.1) hat Bergman einen Integraloperator A durch

$$u = Af = \int_{-1}^{1} g(z, z^*, t) f\left[\frac{z}{2}(1 - t^2)\right](1 - t^2)^{-1/2}\,\mathrm{d}t \tag{2.2}$$

definiert. g heisst die *Erzeugende* von A und hängt nur von $b(z, z^*)$ und $c(z, z^*)$ ab, aber nicht von der speziellen Wahl der analytischen Funktion f. Letztere heisst die *Zugeordnete* von $u = Af$.

A heisst ein *Operator* 1. *Art*, wenn g den Bedingungen

$$g(0, z^*, t) = 1, \quad g(z, 0, t) = 1 \quad (-1 \leqslant t \leqslant 1) \tag{2.3}$$

genügt. Dann hat A eine besonders einfache Inverse.

Die Erzeugende $g(z, z^*, t)$ in (2.2) erhält man als Lösung der Gleichung

$$(1 - t^2)g_{z^*t} - t^{-1} g_{z^*} + 2z\,tLg = 0. \tag{2.4}$$

Einen Operator 1. Art liefert der Ansatz

$$g(z, z^*, t) = 1 + \sum_{n=1}^{\infty} t^{2n} z^n r_{2n}(z, z^*), \quad r_{2n}(z, 0) = 0. \tag{2.5}$$

Durch Einsetzen in (2.4) ergibt sich die einfache Rekursion

$$r_{2,z^*} = -2c, \quad r_{2n+2,z^*} = -\frac{2}{2n+1} Lr_{2n}, \quad (n = 1, 2, \ldots). \tag{2.6}$$

Setzen wir nun

$$r_{2n}(z, z^*) = \int_0^{z^*} p_{2n}(z, \xi)\,\mathrm{d}\xi,$$

so folgt aus (2.6) die von Bergman in [1] angegebene Rekursion

$$p_2 = -2c,$$

(2.7)
$$p_{2n+2} = -\frac{2}{2n+1}\left(p_{2n,z} + bp_{2n} + c\int_0^{z^*} p_{2n}(z, \xi)d\xi\right).$$

Die Konvergenz von (2.5) lässt sich z. B. mit Hilfe der Dominantenmethode nachweisen, wie Bergman [2] gezeigt hat. Vgl. auch Eichler [6].

3. Exponentialoperatoren

Ein Bergman-Operator (2.2) mit einer Erzeugenden der Form

(3.1)
$$g(z, z^*, t) = \exp\sum_{\mu=0}^{m} q_\mu(z, z^*)t^\mu$$

heisst ein *Exponentialoperator*. L in (2.1) heisse *von der Klasse* $\mathcal{E}$ wenn sich die Lösungen von (2.1) mittels eines Exponentialoperators darstellen lassen.

Wie in [9] gezeigt wurde, haben Exponentialoperatoren A die Eigenschaft, dass Lösungen $u = u_n = Az^n$, $n=0, 1, \ldots$, von (2.1) (mit $L \in \mathcal{E}$) auch noch einer gewöhnlichen linearen homogenen Differentialgleichung genügen. Damit ist man dann in der Lage, die relativ weit entwickelte Theorie der zuletzt genannten Gleichungen im Komplexen bei der Untersuchung von Lösungen partieller Differentialgleichungen heranzuziehen.

Spezielle Exponentialoperatoren findet man schon in [2] und einige Versuche, die entsprechenden Gleichungen (2.1) zu kennzeichnen, in [11] und [12]. Der allgemeine Operator mit g gemäss (3.1) wurde in [8] eingeführt. Dort wurden auch notwendige und hinreichende Bedingungen für die Zugehörigkeit von L zur Klasse $\mathcal{E}$ angeben, also die Klasse $\mathcal{E}$ vollständig gekennzeichnet.

Eine Weiterentwicklung dieser Theorie im Zusammenhang mit der Tricomi-Gleichung stammt von H. Florian [7]. Vgl. auch K. Ecker und H. Florian [5] sowie E. Lanckau [10].

4. Minimale Exponentialoperatoren

Bei Anwendungen der Bergman-Operatoren ist es wesentlich, zu einer gegebenen Gleichung (2.1) eine möglichst einfache Erzeugende in (2.2) zu wählen, damit sich die Eigenschaften der Zugeordneten f in einfacher Weise in Eigenschaften der Lösung $u = Af$ transformieren.

Wir wollen nun zeigen, dass man bei Exponentialoperatoren entsprechende Möglichkeiten hat. Es wird sich nämlich ergeben, dass die Klasse $\mathcal{E}$ in zwei disjunkte Teilklassen zerfällt, die wir mit $\mathcal{E}_A$ und $\mathcal{E}_B$ bezeichnen, und dass zu gegebenem $L \in \mathcal{E}_A$ oder $\in \mathcal{E}_B$ ein eindeutig bestimmter „einfachster" Exponentialoperator existiert, dessen Erzeugende sich genau angeben lässt. Wir definieren hierzu:

Ein Exponentialoperator heisse *minimal bezüglich eines gegebenen Operators L in* (2.1), wenn es für dieses L keinen Exponentialoperator gibt, dessen Erzeugende im Exponenten weniger nicht identisch verschwindende Glieder als diejenige von A besitzt.

Um die weiteren Ergebnisse einfach formulieren zu können, schreiben wir [vgl. (3.1)]

$$(4.1)\qquad \begin{cases} g(z,\ z^*,\ t) = \exp q(z,\ z^*,\ t), \\ q(z,\ z^*,\ t) = q_I(z,\ z^*,\ t) + q_{II}(z,\ z^*,\ t) \end{cases}$$

mit

$$(4.2)\qquad q_I(z,\ z^*,\ t) = \sum_{\mu=i}^{m_I} q_{2\mu-1}(z,\ z^*)\, t^{2\mu-1}$$

und

$$(4.3)\qquad q_{II}(z,\ z^*,\ t) = \sum_{\mu=1}^{m_{II}} q_{2\mu}(z,\ z^*)\, t^{2\mu}.$$

Es gilt dann der folgende

S a t z 1. I. *Es ist $L \in \mathcal{C}$ genau dann, wenn einer der beiden folgenden Fälle (A) oder (B) gilt:*

(A) *b in (2.1) ist unabhängig von z^*, und c lässt sich in der folgenden Form darstellen:*

$$(4.4)\qquad c(z,\ z^*) = -\frac{1}{2}\, a_0(z^*) \sum_{\nu=0}^{m_I} a_\nu\, z^\nu, \qquad (a_0 = a_0(z^*),\ a_1, \ldots, a_{m_I}\ \textit{konstant}).$$

(B) *c in* (2.1) *ist unabhängig von z, und b lässt sich in der Form*

$$(4.5\,\mathrm{a})\qquad b(z,\ z^*) = b_1(z) + b_2(z^*)$$

mit beliebigem $b_1(z)$ und

$$(4.5\,\mathrm{b})\qquad b_2(z^*) = 2\int_0^{z^*} c(\xi)\, \mathrm{d}\xi$$

darstellen.

II. *Die Fälle (A) und (B) sind disjunkt, definieren also eine Zerlegung $\mathcal{C} = \mathcal{C}_A \cup \mathcal{C}_B$. Für $L \in \mathcal{C}_A$ hat q_I die Koefizienten*

$$(4.6\,\mathrm{a})\qquad q_{2\mu-1}(z,\ z^*) = \sum_{\nu=\mu-1}^{m_I} k_{\mu\nu}\, a_\nu\, z^{\nu+1/2}, \quad (\mu = 1,\ 2, \ldots)$$

mit

$$(4.6\,\mathrm{b})\qquad k_{1\nu} = 1,\ k_{\mu\nu} = \frac{(-2)^{\mu-1}\, \nu(\nu-1)\ldots(\nu-\mu+2)}{3.5\ldots(2\mu-1)}, \quad (\mu = 2,\ 3, \ldots).$$

Für $L \in \mathcal{C}_B$ ist $q_I \equiv 0$. Für $L \in \mathcal{C}$ hat q_{II} stets die Darstellung

$$q_{II}(z, z^*, t) = -\int_0^z b(\xi)\,d\xi + \sum_{\nu=1}^{m} c_\nu z^\nu (1-t^2)^\nu . \tag{4.7}$$

Dabei sind für $L \in \mathcal{C}_A$ die c_ν beliebige Konstanten. Für $L \in \mathcal{C}_B$ ist $c_1 = -b_2(z^*)$ und die übrigen c_ν sind beliebige Konstanten.

III. *Der minimale Exponentialoperator bezüglich eines $L \in \mathcal{C}_A$ hat die Erzeugende mit dem Exponenten*

$$q_I(z, z^*, t) - \int_0^z b(\xi)\,d\xi$$

[*vgl.* (4.2)], *und bezuglich eines $L \in \mathcal{C}_B$ hat er die Erzeugende mit dem Exponenten*

$$-\int_0^z b(\xi)\,d\xi + b_2(z^*)\,z\,t^2.$$

B e w e i s. Setzt man (3.1) in (2.4) ein und dann den Koeffizienten jeder vorkommenden Potenz von t gleich 0, so erhält man ein System linearer partieller Differentialgleichungen 2. Ordnung, das nach der in [8, S. 915—919], angegebenen Methode behandelt werden kann. Dabei gewinnt man Darstellungen für die Koeffizienten q_μ in (3.1) und Bedingungen hinsichtlich der möglichen Form der Koeffizienten $b(z, z^*)$ und $c(z, z^*)$ in (2.1). Aus (3.4) ...(3.9) in [8] erhalt man dann ohne Mühe (4.4)...(4.6) der vorliegenden Arbeit. Dass $\mathcal{C}_A$ und $\mathcal{C}_B$ disjunkt sind, ist ebenfalls klar : Es sei $L \in \mathcal{C}_A$. Dann kann b nicht von z^* abhängen. Da der Trivialfall $c \equiv 0$ stets ausgeschlossen ist, hängt b in (4.5 a) aber stets von z^* ab, und so folgt aus $L \in \mathcal{C}_A$ nun $L \notin \mathcal{C}_B$. Ist andererseits $L \in \mathcal{C}_B$, so hängt b von z^* ab, also folgt $L \in \mathcal{C}_A$.

Wir beweisen nun die Darstellung (4.7). Die obengenannte Methode in [8] liefert für unsere Darstellung (3.1) die Koeffizienten $q_{2\mu}$ in der Form

$$q_2 = \sum_{\nu=1}^{[m/2]} l_\nu z^\nu \tag{4.8}$$

und

$$q_{2\mu} = \frac{(-1)^{\mu+1}}{\mu!} \sum_{\nu=\mu}^{[m/2]} (\nu-1)(\nu-2)\dots(\nu-\mu+1)\, l_\nu z^\nu, \tag{4.9}$$

$$(\mu = 2, 3, \dots).$$

Weiterhin gilt

$$b(z) = -q_0'(z) - \frac{q_2}{z}. \tag{4.10}$$

Hieraus folgt nun zunächst

$$q_0(z) = -\int_0^z b(\xi)\,d\xi + \sum_{r=0}^{[m/2]} c_r\, z^r \quad \left(c_r = -\frac{l_r}{r}\right).$$

Dabei können wir $[m/2]$ durch eine andere natürliche Zahl m_{II} ersetzen, da die Koeffizienten ungerader Potenzen von t (für die m_I wegen (4.4) durch c bestimmt ist) nicht mit den Koeffizienten gerader Potenzen gekoppelt sind. So erhalten wir insgesamt wegen (4.8)...(4.10) unter Vertauschung der Summationsreihenfolge

$$q_{II} = \sum_{\mu=0}^{m_{II}} q_{2\mu}\, t^{2\mu} = -\int_0^z b(\xi)\,d\xi +$$

$$+ \sum_{\nu=0}^{m_{II}} \sum_{\mu=0}^{\nu} \frac{(-1)^{\mu+1}}{\mu\,!} (\nu-1)(\nu-2)\dots(\nu-\mu+1) l_\nu z^\nu t^{2\mu} =$$

$$= -\int_0^z b(\xi)\,d\xi + \sum_{\nu=0}^{m_{II}} c_\nu z^\nu \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} (-t^2)^\mu =$$

$$= -\int_0^z b(\xi)d\xi + \sum_{\nu=1}^{m_{II}} c_\nu\, [z(1-t^2)]^\nu\ .$$

Dass c_1 im Fall (B) von z^* abhängen muss, ergibt sich aus [8], denn dort wurde bewiesen, dass im Fall (B)

$$c = -\frac{q_{2,z^*}}{2z} \tag{4.11}$$

ist.

Wir beweisen nun die Aussage III. Da sich q_{II} gemäss (4.7) als eine Funktion von $\varphi(z, t) = z(1-t^2)$ darstellen lässt und f in (2.2) ebenfalls von dieser Variablen abhängt und da weiterhin f beliebig gewählt werden kann, so ist mit $\exp q$ auch $\exp\left(q_I - \int_0^z b(\xi)\,d\xi\right)$ eine Erzeugende für das jeweils vorliegende $L \in \mathcal{C}$. Wir können also im Falle (A) ohne Beschränkung der Allgemeinheit bezüglich b und c in (2.2) von q zu $q_I - \int_0^z b(\xi)\,d\xi$ übergehen. Entsprechend kann man sich im Fall (B) auf $m_{II} = 1$ beschränken: Man hat dann wieder eine Erzeugende für dieselbe Gleichung (2.1).

Dass man b und c damit tatsächlich keine neuen zusätzlichen Bedingungen auferlegt, ersieht man aus (4.11) und (4.10). Potenzen von z, die in q_2 bei $m_{II} > 1$ auftreten würden, kann man zu $q_0'(z)$ in (4.10) hinzunehmen, so dass man nichts verliert. Die so gewonnenen Operatoren sind minimal, denn die Koeffizienten von L bestimmen die a_ν und m_I in (4.4) sowie das erste Glied auf der rechten Seite von (4.7) und für $L \in \mathcal{C}_B$ auch c_1 in (4.7) eindeutig. Damit bleibt kein Spielraum für weitere Reduktionen. Dies war zu zeigen.

Aus Satz 1 folgt auch, dass sich (2.1) mit $L \in \mathcal{C}_B$ in der Form

$$u_{zz^*} + bu_{z^*} + \frac{1}{2} b_{z^*} u = 0,$$

oder

$$u_{zz} + \sqrt{b}\,(\sqrt{b}\,u)_{z^*} = 0$$

darstellen lässt.

5. Entwicklung nach $\{Az^n\}$

Für $L \in \mathcal{C}$ lasst sich $u = Af$ in eine Reihe nach $u_n = Az^n$, $n = 0, 1, \ldots,$ entwickeln. Es gilt der

S a t z 2. *Es sei $L \in C$ und $b(z, z^*)$ und $c(z\, z^*)$ seien ganze Funktionen. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ konvergiere für $|z| < R$ $(R > 0)$. Dann konvergiert*

$$u(z, z^*) = \sum_{n=0}^{\infty} a_n u_n(z, z^*), \quad (u_n = Az^n) \tag{5.1}$$

fur $|z| < R$ und jedes endliche z^*, und es gilt für die genannten (z, z^*) asymptotisch bezüglich n

$$u_n \sim e^{q_0(z)} z^n \sqrt{\pi}\, \frac{\Gamma\left(n + \frac{1}{2}\right)}{2^n\, n!}. \tag{5.2}$$

B e w e i s. b und c sind in jedem beschränkten Bereich B des zz^*-Raumes beschränkt. Wegen Satz 1 ist also der Exponent $q(z, z^*, t)$ in (3.1) für $(z, z^*) \in B$, $t \in [-1, 1]$ beschränkt. Dasselbe gilt demnach auch für $g(z, z^*, t)$. Wegen

$$\int_{-1}^{1} (1 - t^2)^{n-1/2}\, dt = \frac{\sqrt{\pi}\,\Gamma\left(n + \frac{1}{2}\right)}{n!} < \sqrt{\pi}$$

und (2.2) folgt damit die Aussage über die Konvergenz von (5.1). Weiterhin gibt es zu jedem $(z, z^*) \in B$ eine Konstante k derart, dass

$$|\, g(z,\ z^*,\ t) - e^{q_0(z)} \,| \leqslant k\,|t|, \quad (z,\ z^*) \in B, \quad t \in [-1,\ 1]$$

gilt. Dann wird

$$\left| u_n - e^{q_0(z)}\, 2^{-n} z^n \int_{-1}^{1} (1-t^2)^{n-1/2}\, dt \right| \leqslant k \int_{-1}^{1} |t|\, (1-t^2)^{n-1/2}\, dt = \frac{k}{n+1/2}$$

und (5.2) ist bewiesen.

6. Beziehungen zwischen Exponentialoperatoren und Bergman-Operatoren 1. Art

Für die Helmholtz-Gleichung $\Delta\psi + k^2\psi = 0$ oder

$$u_{zz^*} + \frac{1}{4} u = 0 \tag{6.1}$$

ist $L = \partial^2/\partial z\, \partial z^* + 1/4 \in \mathcal{O}_A$, wie aus Satz 1 folgt. Mögliche Exponentialerzeugende sind

$$g(z,\ z^*,\ t) = \exp\left[i\sqrt{zz^*}\, t + \sum_{\nu=1}^{m_{II}} c_\nu\, z^\nu\, (1-t^2)^\nu \right]. \tag{6.2}$$

Die einfachste ist also

$$g_0(z,\ z^*,\ t) = \exp i\sqrt{zz^*}\, t. \tag{6.3}$$

Der zugehörige Operator, den wir mit A_0 bezeichnen, liefert für $f(z) = z^n$ die Lösungen

$$u_n(r,\ \theta) = A_0 z^n = \sqrt{\pi}\, \Gamma\left(n + \frac{1}{2}\right) e^{in\theta}\, J_n(r),$$

wobei $z = re^{i\theta}, z^* = re^{-i\theta}$ gesetzt wurde. Dies folgt aus (2.2) und einer bekannten Integraldarstellung der Bessel-Funktionen 1. Art.

Wie man sieht, kommt man in (6.2) im vorliegenden Falle mit $m_I = 1$ aus. Dies trifft nicht nur für die relativ einfache Gleichung (6.1) zu, sondern auch für kompliziertere $L \in \mathcal{O}_A$. Aus Satz 1 folgt nämlich:

Es ist $L \in \mathcal{O}_A$ mit $m_I = 1$ genau dann, wenn b nicht von z^* und c nicht von z abhängt. In (3.1) *ist dann*

$$\begin{cases} q_0(z) = -\displaystyle\int_0^z b(\xi)\, d\xi, \\[2ex] q_1(z,\ z^*) = 2i \left(z \displaystyle\int_0^{z^*} c(\xi)\, d\xi \right)^{1/2}. \end{cases} \tag{6.4}$$

Für (6.1) liefert (2.5)—(2.6) einfach

$$g(z,\ z^*,\ t) = \cos\sqrt{zz^*}\,t. \tag{6.5}$$

Dies lässt erkennen, dass in gewissen Fällen Beziehungen zwischen Bergman-Operatoren 1. Art, deren Erzeugende man auf die in Abschnitt **2** angegebene Weise gewinnt, und Exponentialoperatoren zu erwarten sind. Diese Zusammenhänge sollen nun erörtert werden.

Aus (2.4) folgt, dass mit $g(z,\ z^*,\ t)$ auch $g(z,\ z^*,\ -t)$ eine Lösung ist. Also ist für $L \in \mathcal{C}$ mit (4.1) auch

$$\tilde{g}(z,\ z^*,\ t) = \exp q_{II}(z,\ z^*,\ t)\cos q_I(z,\ z^*,\ t)$$

eine Erzeugende für das betreffende L, und für $L \in \mathcal{C}_A$ ist die einfachste Erzeugende

$$\tilde{g}_0(z,\ z^*,\ t) = \cos q_I(z,\ z^*,\ t). \tag{6.6}$$

(6.3) und (6.5) bieten ein Beispiel dafür.

Umgekehrt kann man die Rekursion in Abschnitt **2** so ändern, dass sie unmittelbar auf Exponentialerzeugende führt. Bemerkenswert ist, dass sich diese Modifikation ohne Komplizierung der Rekursionsformeln ermöglichen lässt: Dazu wähle man den Ansatz

$$g(z,\ z^*,\ t) = 1 + \sum_{n=1}^{\infty} t^n\, z^{n/2} \int^{z^*} p_n(z,\ \xi)\,d\xi. \tag{6.7}$$

Dann liefert (2.4) einfach

$$\begin{cases} p_2 = -2c, \\ p_{n+2} = -\dfrac{2}{n+1}\left(p_{n,z} + bp_n + c\displaystyle\int_0^{z^*} p_n(z,\ \xi)\,d\xi\right), \end{cases} \tag{6.8}$$

wobei $p_1(z,\ z^*)$ noch frei wählbar ist. Der Konvergenzbeweis ist fast wörtlich derselbe wie in Bergman [1].

Zum Beispiel erhalten wir für (6.1) die Erzeugende (6.3), wenn wir $p_1 = i2^{-1}z^{*-1/2}$ wählen.

Es lassen sich nun Klassen von Gleichungen (2.1) angeben, für die (6.7) auf Exponentialerzeugende führt. Es gilt der

Satz 3. *Für jede Differentialgleichung der Form*

$$u_{zz^*} + c(z^*)\,u = 0\,, \tag{6.9}$$

oder

$$u_{zz^*} + k(z^*)\, u_{z^*} + \frac{1}{2} k'(z^*)\, u = 0 \tag{6.10}$$

liefert (6.7) *bei geeigneter Wahl von* $p_1(z, z^*)$ *dieselbe Exponentialerzeugende wie Satz 1, III, nämlich*

$$g(z, z^*, t) = \exp\left(\mathrm{i}c_0(z^*)\sqrt{z}\, t\right) \tag{6.11}$$

mit

$$c_0(z) = 2\sqrt{\int_0^{z^*} c(\xi)\, \mathrm{d}\xi} \tag{6.12}$$

bzw.

$$g(z, z^*, t) = \exp\left[-k(z^*) z t^2\right]. \tag{6.13}$$

B e w e i s. Wir betrachten (6.9). Aus (6.8) und (6.12) folgt $p_2 = c_0 c_0' \cdot$ p_1 ist willkürlich, und wir setzen $p_1 = \mathrm{i}c_0'$. Da p_1 und p_2 nicht von z abhängen und $b \equiv 0$ ist, so folgt aus (6.8), dass p_n für alle n nicht von z abhängt. Demnach wird

$$p_{n+2} = -\frac{c_0 c_0'}{n+1} \int_0^{z^*} p_n(\xi)\, \mathrm{d}\xi.$$

Daraus folgt (6.11) induktiv. Entsprechend schliesst man bei (6.10). Dabei ist $p_1 \equiv 0$ zu setzen. Aus (6.8) folgt dann $p_{2n+1} \equiv 0$ $(n = 2, 3, \ldots)$.

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ASUPRA TEORIEI OPERATORILOR EXPONENŢIALI ŞI A OPERATORILOR BERGMAN DE SPECIA I

(Rezumat)

Operatorii exponenţiali şi operatorii Bergman de specia I-a, introduşi pentru ecuaţiile cu derivate parţiale în două variabile independente, ridică probleme cu totul diferite. În lucrare se arată că în numeroase cazuri, între cele două tipuri de operatori există relaţii importante.

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ON THE NATURAL BOUNDARY CONDITIONS FOR MANGERON EQUATIONS

by

SUNDARAM EASWARAN

Dedicated to Professor M. N. OĞUZTÖRELI on his 50th birthday

1. In this paper we investigate some of the properties of the functions $v(x, y)$ minimizing the functional $\int_a^b \int_c^d v_{xy}^2(x, y)\, dx\, dy \Big/ \int_a^b \int_c^d v^2(x, y)\, dx\, dy$ in the space of functions which are continuous with continuous $v_x, v_y, v_{xy}, v_{xyy}, v_{xxy}\ v_{xyxy}$, if the rectangle $a \leqslant x \leqslant b,\ c \leqslant y \leqslant d$ and obtain, the boundary conditions satisfied by the minimizing functions.

2. In [2] it has been proved that if a function $u(x) \in c^2(a, b)$ minimizes the expression

$$\int_a^b y'^2(x)\, dx \Big/ \int_a^b y^2(x)\, dx, \tag{1}$$

then $u(x)$ satisfies the equation

$$u''(x) + \lambda u(x) = 0 \tag{2}$$

and the boundary conditions

$$u'(a) = 0, \quad u'(b) = 0, \tag{3}$$

where λ is the minimum of expression (1).

The boundary conditions (3) are called natural boundary conditions. In this Note it is our aim to give an analogous result in two dimensions which is related to Mangeron Equations. The expression to be minimized in our case is

$$\int_a^b \int_c^d v_{xy}^2(x, y)\, dx\, dy \Big/ \int_a^b \int_c^d v^2(x, y)\, dx\, dy. \tag{4}$$

The results can be generalized to n-dimensions, but for the sake of simplicity, we will consider only the two dimensional problem.

2. Let R denote the rectangle $\{a \leqslant x \leqslant b\,;\ c \leqslant y \leqslant d\}$. By $F^{(0)}$ we denote the space of all continuous functions defined on R with the inner product

$$(u, v) = \int_a^b \int_c^d u(x,y)\, v(x, y)\, dx\, dy, \quad u \in F^{(0)}, \quad v \in F^{(0)}. \tag{5}$$

By $F^{(\infty)}$ we denote the space of infinitely differentiable function defined on R with the norm defined by (5). It can be shown that $F^{(\infty)}$ is dense in $F^{(0)}$. Let Γ' denote the class of all continuous functions $u(x, y)$ defined on R such that $u(x, y)$, $u_y(x, y)$ and $u_{xy}(x, y)$ are all continuous in R. By Γ'' we will denote the class of all continuous functions on R such that $u(x, y)$ and $u_{xy}(x, y) \in \Gamma''$. Further we introduce the notation

$$[u]_{a,\,c}^{b,\,d} = u(b, d) - u(a, d) - u(b, c) + u(a, c), \tag{6}$$

which shows a double difference operation. In what follows we will consider the problem of minimizing the expression (4) in the class Γ''. We have the following theorem in this connection.

T h e o r e m. *If $u(x, y) \in \Gamma''$ minimizes* (2) *and λ denotes the corresponding minimum then $u(x, y)$ satisfies the Mangeron equation*

$$u_{xyxy}(x, y) - \lambda u(x, y) = 0 \tag{7}$$

and the conditions

$$u_{xy}(a, y) = u_{xy}(b, y) = u_{xy}(x, c) = u_{xy}(x, d) = 0. \tag{8}$$

P r o o f. Let $u \in \Gamma''$ minimize the functional (4) and let λ be the minimum. Then we have

$$\int_a^b \int_c^d [u_{xy}^2(x, y) - \lambda u^2(x, y)]\, dx\, dy = 0.$$

If $v(x, y) \in \Gamma''$ then $u(x, y) + \varepsilon\, v(x, y) \in \Gamma''$ where ε is an arbitrary constant. Then the function $f(\varepsilon)$ defined by

$$f(\varepsilon) = \int_a^b \int_c^d \{(u_{xy}(x, y) + \varepsilon\, v_{xy}(x, y))^2 - \lambda(u(x, y) + \varepsilon\, v(x, y))^2\}\, dx\, d_y \tag{9}$$

attains its minimum at $\varepsilon = 0$. Hence

$$\frac{\mathrm{d}f(\varepsilon)}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = 0.$$

This implies

$$\int_a^b\int_c^d [u_{xy}(x, y)\, v_{xy}(x, y) - \lambda u(x, y)\, v(x, y)]\, \mathrm{d}x\, \mathrm{d}y = 0 \tag{10}$$

for all $v(x, y) \in \Gamma''$. On the other hand we have the following identity

$$\begin{aligned} u_{xy}(x, y)\, v_{xy}(x, y) = &- (v(x, y)\, u_{xy}(x, y))_{xy} + (v_x(x, y)\, u_{xy}(x, y))_y + \\ &+ (v_y(x, y)\, u_{xy}(x, y))_x . \end{aligned} \tag{11}$$

Thus by integrating (10) and by (9) we get

$$\begin{aligned} &[-vu_{xy}]_{a,\,c}^{b,\,d} + \int_a^b [v_x\, u_{xy}(x, d) - v_x\, u_{xy}(x, c)]\, \mathrm{d}x + \\ &+ \int_c^d [v_y\, u_{xy}(b, y) - v_y\, u_{xy}(a, y)]\, \mathrm{d}y + \\ &+ \int_a^b\int_c^d [u_{xyxy}(x, y) - \lambda u(x, y)]\, v(x, y)\, \mathrm{d}x\, \mathrm{d}y = 0. \end{aligned} \tag{12}$$

But the set of all functions in Γ'' satisfying

$$v(x, c) = v(x, d) = v(a, y) = v(b, y) = 0 \tag{13}$$

is dense in $F^{(0)}$. This fact can be easily proved [6]. Hence for this dense set of functions we have

$$\int_a^b\int_c^d [u_{xyxy}(x, y) - \lambda u(x, y)]\, v(x, y)\, \mathrm{d}x\, \mathrm{d}y = 0. \tag{14}$$

Thus the function $u_{xyxy}(x, y) - u(x, y) \in F^{(0)}$ must be identically zero. Thus we have

$$u_{xyxy}(x, y) - u(x, y) = 0. \tag{7'}$$

This proves (7). Now going back to (12) we get

$$
(15) \qquad \begin{aligned} &- [v u_{xy}]_{a,c}^{b,d} + \int_a^b [v_x u_{xy}(x, d) - v_x u_{xy}(x, c)] \, dx + \\ &+ \int_c^d [v_y u_{xy}(b, y) - v_y u_{xy}(a, y)] \, dy = 0 \end{aligned}
$$

for all $v(x, y) \in \Gamma''$.

Now let $f(x)$ and $g(y)$ be two c^2 functions in the respective domains such that

$$
(16) \qquad f(x) \not\equiv 0, \quad g(y) \not\equiv 0; \quad f(a) = f(b) = 0, \quad g(c) = 0, \quad g(d) \neq 0.
$$

Let

$$
(17) \qquad \begin{cases} v(x, y) = f(x) g(y); \quad \text{then } v(x, y) \in \Gamma'' \text{ and} \\ v(a, c) = v(b, d) = v(b, c) = v(a, d) = 0, \\ v(x, c) = 0, \; v(a, y) = 0, \quad v(b, y) = 0. \end{cases}
$$

Then (17) reduces to

$$
(18) \qquad g(d) \int_a^b f'(x) u_{xy}(x, d) \, dx = 0.
$$

Thus for al $f(x) \in c''$ and $f(a) = 0 = f(b)$ we have

$$
(19) \qquad \int_a^b f'(x) u_{xy}(x, d) \, dx = 0.
$$

Hence by the fundamental lemma of Calculus of Variations we get

$$
u_{xy}(x, d) = \alpha_4 \text{ (a constant).}
$$

Similarly we have

$$
u_{xy}(a, y) = \alpha_1, \quad u_{xy}(b, y) = \alpha_2, \quad u_{xy}(x, c) = \alpha_3.
$$

Therefore

$$
(20) \qquad \begin{cases} u_{xy}(a, d) = \alpha_1 = \alpha_4, \quad u_{xy}(a, c) = \alpha_1 = \alpha_3, \\ u_{xy}(b, c) = \alpha_2 = \alpha_3, \quad u_{xy}(b, d) = \alpha_4 = \alpha_2. \end{cases}
$$

Thus we have

$$
(21) \qquad u_{xy}(a, y) = u_{xy}(b, y) = u_{xy}(x, c) = u_{xy}(x, d) = c,
$$

where c is a constant. Substituting these boundary values of $u_{xy}(x, y)$ in (15) we get

$$c\,[v]_{a,c}^{b,d} = 0, \quad \text{for all} \quad v(x, y) \in \Gamma'',$$

which implies that $c = 0$ because of the arbitrariness of v. Hence the boundary conditions (8) are all satisfied. This completes the proof of the theorem.

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ASUPRA CONDIŢIILOR LA LIMITĂ NATURALE PENTRU ECUAŢIILLE LUI MANGERON

(Rezumat)

Se cercetează unele proprietăţi ale funcţiilor $v(x, y)$ care fac minimă funcţionala $\int_a^d \int_c v_{xy}^2(x, y)\, dx\, dy \Big/ \int_a^b \int_c^d v^2(x, y)\, dx\, dy$ în spaţiul funcţiilor continue şi cu derivatele v_x, v_y, v_{xy}, v_{xyy}, v_{xxy}, v_{xyxy} în dreptunghiul $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ şi se obţin condiţiile la limită pe care le verifică respectivele funcţii.

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MATHEMATICAL SYSTEMS WITH MIXED STRUCTURES

SOLUTIONS OF CERTAIN FUNCTIONAL INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS [1])

BY

MEHMET NAMIK OĞUZTÖRELI

Dedicated to Professor D. MANGERON on his 65 th anniversary

1. Recently, D. Mangeron and the author investigated in this journal and in the other periodicals several mathematical systems wich mixed structures [2]...[10] [2]). In the present note we consider certain functional integral and integro-differential equations related with the classical functional equation of Jensen [1].

We begin with the following functional integral equation

$$2F\left(\frac{x+y}{2}\right) = F(x) + F(y) + \tag{1}$$
$$+ \lambda\left\{\int_0^x K(x-\xi)\, F(\xi)\, d\xi + \int_0^y K(y-\eta)\, F(\eta)\, d\eta\right\},$$

which reduces to the Jensen's functional equation for $\lambda = 0$. Here x and y are real variables, λ is a real parameter and $K(x)$ is a continuous function in a neighborhood of $x = 0$. We assume that the functions

[1]) This work was partly supported by the National Research Council of Canada under the Grant NRC-A 4345 through the University of Alberta.

[2]) The reader is invited to consult both our joint with Prof. D. Mangeron paper as well as the author's paper of a very recent publication in the *Rendiconti dell' Accademia di Scienze Fisiche e Matematiche di Napoli.*

$$\int_0^x K(x-\xi)\,d\xi, \qquad \int_0^x \xi K(x-\xi)\,d\xi \tag{2}$$

are linearly independent. Under these conditions we wish to find real valued continuous solutions of Eq. (1) in some neighborhood of $x = 0$.

Let us note that the functions (2) are linearly independent for a large class of functions $K(x)$.

Now, assuming that $\lambda \neq 0$, we seek a solution to Eq. (1.1) in the form

$$F(x) = \sum_{m=0}^{\infty} F_m(x)\lambda^m, \tag{3}$$

where the functions $F_m(x)$ are supposed to be continuous in some neighborhood of $x = 0$. Substituting the power serios (3) into Eq. (1) and comparing similar terms, we obtain the following integro-functional recursive system :

$$\begin{cases} 2F_0\left(\dfrac{x+y}{2}\right) = F_0(x) + F_0(y), \\ 2F_m\left(\dfrac{x+y}{2}\right) = F_m(x) + F_m(y) + \displaystyle\int_0^x K(x-\xi)F_{m-1}(\xi)\,d\xi + \\ \qquad + \displaystyle\int_0^y K(y-\eta)F_{m-1}(\eta)\,d\eta, \quad (m = 1, 2, 3, \ldots). \end{cases} \tag{4}$$

The first of the above equations is J e n s e n's functional equation and its general solution is of the form

$$F_0(x) = a_0 x + b_0, \tag{5}$$

where a_0 and b_0 are constants. Hence we have

$$\begin{aligned} 2F_1\left(\frac{x+y}{2}\right) = F_1(x) + F_1(y) + a_0\left\{\int_0^x \xi K(x-\xi)\,d\xi + \int_0^y \eta K(y-\eta)\,d\eta\right\} + \\ + b_0\left\{\int_0^x K(x-\xi)\,d\xi + \int_0^y K(y-\eta)\,d\eta\right\}, \end{aligned} \tag{6}$$

and, putting $y = x$, we find

$$a_0 \int_0^x \xi K(x-\xi)\,d\xi + b_0 \int_0^y K(x-\xi)\,d\xi \equiv 0, \tag{7}$$

which, by the linear independence of the functions (2), yields $a_0 = b_0 = 0$. Thus we have

$$F_0(x) \equiv 0, \quad F_1(x) = a_1 x + b_1, \tag{8}$$

where a_1 and b_1 are constants. Repeating the above argument we can show, step by step that all $F_m(x)$'s are identically zero. Consequently $F(x) \equiv 0$ is the only continuous solution of Eq. (1) for $\lambda \neq 0$, and $F(x) = ax + b$ (a, b constants) is the general solution for $\lambda = 0$.

2. We now consider the functional integro-differential equation

$$\begin{aligned} 2F^{(n)}\left(\frac{x+y}{2}\right) = F(x) + F(y) + \lambda \Bigg\{ \int_0^x K(x-\xi) F(\xi)\,d\xi + \\ + \int_0^y K(y-\eta)\,F(\eta)\,d\eta \Bigg\}, \end{aligned} \tag{9}$$

where n is a natural number, $F^{(n)}(x) = \dfrac{d^n F(x)}{dx^n}$ and $K(x)$ is a continuously differentiable function in some neighborhood of $x = 0$.

Clearly, any n-times differentiable solution of Eq (9) is also differentiable of order $n+1$. We now seek a solution of the form (3), where the functions are assumed to be $(n+1)$ - times differentiable. By substitution we find

$$\begin{cases} 2F_0^{(n)}\left(\dfrac{x+y}{2}\right) = F_0(x) + F_0(y), \\[2ex] 2F_m^{(n)}\left(\dfrac{x+y}{2}\right) = F_m(x) + F_m(y) + \displaystyle\int_0^x K(x-\xi) F_{m-1}(\xi)\,d\xi + \\[2ex] \qquad + \displaystyle\int_0^y K(y-\eta) F_{m-1}(\eta)\,d\eta, \qquad (m = 1, 2, 3, \ldots). \end{cases} \tag{10}$$

We now differentiate the first equation in (10) with respect to x, which yields

$$F_0^{(n+1}\left(\frac{x+y}{2}\right) = F_0'(x) \equiv c, \tag{11}$$

where c is a constant, since the left hand side of Eq. (11) depends on y, but the right hand side does not. Hence, $F_0(x) = cx + b$, where b is another constant. We now substitute this expression into the first equation of (10). In this way we find $b = c = 0$, i. e.,

$$F_0(x) \equiv 0. \tag{12}$$

Consequently we have

$$2F_1^{(n)}\left(\frac{x+y}{2}\right) = F_1(x) + F_1(y) \tag{13}$$

by virtue of Eq. (10) and (12). The $n+1$ - times differentiable solution of Eq. (13) is $F_1(x) \equiv 0$, which can be established by the similar argument. Repeating this process we see that all $F_m(x)$ are identically zero. Therefore $F(x) \equiv 0$ is the only n-times differentiable solution of Eq. (9).

3. Using the techniques of the previous sections we can easily show that $F(x) \equiv 0$ is the only n-times differentiable solution of the functional integro-differential equation

$$\begin{aligned} 2F\left(\frac{x+y}{2}\right) &= F^{(n)}(x) + F^{(n)}(y) + \\ &+ \lambda\left\{\int_0^x K(x-\xi)F(\xi)\,d\xi + \int_0^y K(y-\eta)\,F(\eta)\,d\eta\right\}, \end{aligned} \tag{14}$$

where $K(x)$ being as in § **2**.

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SISTEME MATEMATICE CU STRUCTURĂ COMPLEXĂ

Soluţii ale unor ecuaţii funcţionale integrale şi integro-diferenţiale

(Rezumat)

Autorul, urmînd ordinea de idei a unui şir de lucrări elaborate în colaborare cu D. Mangeron la Universitatea din Alberta [2]...[10], studiază soluţiile unor ecuaţii funcţionale integrale şi funcţionale integro-diferenţiale.

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OBSERVAŢII ASUPRA PROBLEMEI CUADRATURILOR

DE

E. ROGAI

1. G. Karapandjitch a dat în [2] o metodă de obţinere a unor primitive (sau a unor cuadraturi după limbajul utilizat în articol) folosind integrale particulare ale ecuaţiei diferenţiale liniare:

$$y' + P(x)y + Q(x) = 0, \quad P(x),\ Q(x) \in C^0(I), \tag{1}$$

iar în [3] a aplicat aceeaşi metodă, în acelaşi scop, ecuaţiei lui Bernoulli de formă particulară

$$y' + P(x)y + Q(x)y^2 = 0, \tag{1'}$$

care, după cum se ştie, este reductibilă la o ecuaţie diferenţială liniară.

Ideea originală şi interesantă, care stă la baza metodei lui Karapandjitch constă [2], [3] în a scrie ecuaţia ce dă integrala generală a lui (1) sub forma:

$$\int Q(x)\, e^{\int P(x)dx} dx = C - y(x) e^{\int P(x)dx}, \tag{2}$$

de a utiliza această relaţie la determinarea unei integrale particulare $y_0(x)$, ce corespunde valorii C_0 a lui C,

$$\int Q(x)\, e^{\int P(x)dx} dx = C_0 - y_0(x) e^{\int P(x)dx} \tag{3}$$

şi de a interpreta relaţia (3) după cum urmează.

Dacă se cunoaşte soluţia particulară $y_0(x)$ a ecuaţiei (1) *ce corespunde lui $C = C_0$ şi o primitivă a lui $P(x)$, atunci se cunoaşte, în virtutea relaţiei* (3), *şi primitiva funcţiei* $Q(x)e^{\int P(x)dx}$.

2. În limbaj propriu, această metodă poate fi prezentată şi altfel, în baza interpretării formulei (2), cunoscută din teoria ecuaţiilor diferenţiale ([1], [4]...[8]).

Astfel

$$e^{-\int P(x)dx} = \bar{y}_0(x) \tag{4}$$

este o soluţie particulară $\bar{y}_0(x)$ a ecuaţiei omogene [7] $\bar{y}' + P(x)\bar{y} = 0$, asociate ecuaţiei (1), iar

$$\int Q(x)\,e^{\int P(x)dx}\,dx = \int \frac{Q(x)}{\bar{y}_0(x)}\,dx \tag{5}$$

reprezintă pe $-C(x)$, unde $C(x)$ este funcţia ce intervine în expresia soluţiei particulare $y_0(x)$ a ecuaţiei (1), $y_0(x) = C(x)\,\bar{y}_0(x)$, soluţie determinată prin metoda lui Lagrange.

Cu aceste precizări, relaţia (3) care leagă soluţiile $\bar{y}_0(x)$ şi $y_0(x)$ cu funcţia $Q(x)$ şi constanta C_0 se poate scrie sub forma :

$$\int \frac{Q(x)}{\bar{y}_0(x)}\,dx = C_0 - \frac{y_0(x)}{\bar{y}_0(x)}. \tag{3'}$$

Avem astfel următorul rezultat :

Dacă se dă integrala generală a ecuaţiei diferenţiale (1), *adică se dau cuadratura din* (4) *şi o soluţie particulară* $y_0(x)$ *a ecuaţiei* (1), *atunci se cunoaşte cuadratura dată de relaţia* (5), *prin intermediul formulei* (3').

Rezultatul exprimă o proprietate inversă (într-un anumit sens) a proprietăţii cunoscute a ecuaţiei diferenţiale liniare de ordinul I neomogenă, anume că pentru a obţine integrala ei generală este necesară efectuarea a două cuadraturi, cea din (4) şi cea dată de relaţia (5).

3. Avînd în vedere că ecuaţia Bernoulli generală

$$y' + P(x)y + Q(x)y^m = 0, \qquad P(x),\ Q(x) \in C_0(I), \qquad m \in R \tag{6}$$

şi cea Riccati

$$y' = p(x)y^2 + q(x)y + r(x), \qquad p(x),\ q(x),\ r(x) \in C_0 \tag{7}$$

(ultima numai cînd se cunoaşte o soluţie particulară $y_1(x)$) sînt reductibile la ecuaţii liniare de forma (1), vom adopta metoda din [2] la aceste ecuaţii.

Pentru ecuaţia Bernoulli, ecuaţia corespunzătoare lui (3) rezultă :

$$\int (1-m)\,Q(x)e^{\int(1-m)P(x)dx}\,dx = C_0 - y_0^{1-m}(x)\,e^{\int(1-m)P(x)dx}. \tag{8}$$

În cazul ecuaţiei Riccati, pentru care admitem că se cunoaşte integrala particulară $y_1(x)$, ecuaţia (3) se scrie :

$$\int p(x)\,e^{\int[2p(x)y_1(x)+q(x)]dx}\,dx = C_0 - \frac{e^{\int[2p(x)y_1(x)+q(x)]dx}}{y_0(x) - y_1(x)}. \tag{9}$$

Deci, *dacă se cunosc pentru ecuaţia* (6) *o soluţie particulară* $y_0(x)$ *(ce corespunde constantei* C_0*) şi o primitivă a lui* $(1-m)\,P(x)$, *atunci cu relaţia* (8) *se poate determina o anumită primitivă a funcţiei* $(1-m)Q(x)e^{\int(1-m)P(x)dx}$.

Similar, *dacă se cunosc pentru ecuaţia* (7) *soluţiile particulare* $y_1(x)$ *şi* $y_0(x)$ *(ultima corespunzînd constantei* C_0*) şi o primitivă a lui* $2p(x)y_1(x) + q(x)$, *atunci cu ajutorului lui* (9) *se poate calcula o anumită primitivă a lui* $p(x)\, e^{\int(2py_1+q)dx}$.

4. Se ştie că ecuaţia diferenţială liniară de ordinul al doilea omogenă

$$y'' + a(x)y' + b(x)y = 0, \tag{10}$$

cu $a(x)$ şi $b(x)$ funcţii de x de anumite clase, ajunge prin transformarea de funcţie $z(x) = \dfrac{y'(x)}{y(x)}$ la următoarea ecuaţie Riccati:

$$z' = -z^2 - a(x)z - b(x). \tag{11}$$

Dacă pentru (11) se cunosc două integrale particulare $z_1(x)$ şi $z_0(x)$, formula (9) permite să scriem:

$$\int p(x)\, e^{\int[2p(x)z_1(x)+q(x)]dx}\, dx = C_0 - \frac{e^{\int[2p(x)z_1(x)+q(x)]dx}}{z_0(x) - z_1(x)}. \tag{12}$$

Notînd cu $y_1(x)$ şi $y_0(x)$ soluţiile ecuaţiei (10), corespunzătoare lui $z_1(x)$ şi $z_0(x)$ şi avînd în vedere că $p(x) = -1$ şi $q(x) = -a(x)$, relaţia (12) devine:

$$\int \frac{e^{-\int a(x)\,dx}}{y_1^2(x)}\, dx = \frac{e^{-\int a(x)dx}}{y_1^2\left[\dfrac{y_0'(x)}{y_0(x)} - \dfrac{y_1'(x)}{y_1(x)}\right]} - C_0, \tag{13}$$

care are următoarea interpretare:

Dacă pentru ecuaţia (10) *se cunosc două integrale particulare distincte* $y_1(x)$ *şi* $y_0(x)$, *atunci prin intermediul lui* (13) *se cunoaşte şi o primitivă a funcţiei* $y_1^{-2}(x)e^{-\int a(x)dx}$.

5. În cele ce urmează se consideră cîteva exemple.

5.1. Fie de calculat integrala:

$$I = \int (2 + \ln x) \ln x\, dx.$$

Asociem acestei integrale ecuaţia diferenţială pentru care $Q(x) = 2 + \ln x$, iar $P(x)$ se calculează din $e^{\int P(x)dx} = |\ln x|$, adică ecuaţia:

$$y' + \frac{y}{x \ln x} + 2 + \ln x = 0,$$

care admite soluţia particulară $y_0(x) = -x \ln x$. Cu ajutorul ei relaţia (3) dă:

$$I = C_0 + x \ln^2 x.$$

5.2. Pentru calculul integralei

$$I = \int e^{ax} \cos bx \, dx,$$

i se asociază ecuaţia diferenţială $y' + ay + \cos bx = 0$. O soluţie particulară a ei are forma $y_0(x) = A \sin bx + B \cos bx$, anume $y_0(x) = -(b \sin bx + a \cos bx)/(a^2 + b^2)$, $C_0 = 0$. Deci, cu ajutorul lui (3), găsim

$$I = \frac{a \cos bx + b \sin bx}{a^2 + b^2} e^{ax}.$$

5.3. Fie ecuaţia Bernoulli

$$y' + \frac{1}{x} y = y^2 \frac{\ln x}{x}, \qquad x > 0.$$

Pentru ea

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{-\ln x}{x}, \quad m = 2, \quad \int (1 - m) P(x) \, dx = \ln \frac{1}{x}$$

şi formula (8) se scrie

$$\int \frac{\ln x}{x} e^{-\frac{dx}{x}} dx = C_0 - y_0^{-1}(x) e^{\int -\frac{dx}{x}}.$$

Cu soluţia particulară $y_0(x) = 1/(1 + \ln x)$ relaţia (8) furnizează

$$\int \frac{\ln x}{x^2} dx = C_0 - \frac{1 + \ln x}{x}. \tag{14}$$

De fapt această soluţie particulară corespunde lui $C_0 = 0$, dacă avem în vedere că integrala generală a ecuaţiei considerate este $y(x) = 1/(1 + Cx + \ln x)$. Cu această precizare, relaţia (14) devine:

$$\int \frac{\ln x}{x^2} dx = -\frac{1 + \ln x}{x}. \tag{15}$$

În conformitate cu [1], obţinem aceeaşi primitivă dată de (15), oricare ar fi integrala particulară $y_0(x)$ folosită.

5.4. Fie şi ecuaţia Riccati

$$y' = \frac{\sin x + \cos x}{\sin x - \cos x} y^2 - \frac{2 \sin x}{\sin x - \cos x} y + 1,$$

pentru care se cunosc două integrale particulare: $y_1(x) = \operatorname{tg} x$ şi $y_0(x) = (\sin x + e^x)/(\cos x + e^x)$. Avem

$$\int [2y_1(x) p(x) + q(x)] \, dx = x + \ln (\operatorname{tg} x - 1) \sqrt{1 + \operatorname{tg}^2 x},$$

respectiv

$$e^{\int [2y_1(x) q(x) + q(x)] dx} = \frac{e^x (\operatorname{tg} x - 1)}{\cos x}.$$

Acum formula (9), cu $y_0(x)$ menţionat, dă

$$\int e^x \frac{\sin x + \cos x}{\cos^2 x} dx = C_0 + \frac{e^x + \cos x}{\cos x}. \tag{16}$$

5.5. Fie ecuaţia diferenţială

$$(1 - x) y'' + xy' - y = 0, \quad x > 1,$$

care are integralele particulare $y(x) = x$ şi $y(x) = e^x$. Să luăm $y_0(x) = e^x$ şi $y_1(x) = x$. Atunci (13) dă

$$\int \frac{(x-1)e}{x^2} dx = -C_0 + \frac{e^x}{x}. \tag{17}$$

Dacă luăm $y_0(x) = x$ şi $y_1(x) = e^x$, obţinem

$$\int (x-1) e^{-x} dx = -C_0 - xe^{-x}.$$

5.6. Fie şi ecuaţia diferenţială [7]

$$xy'' - (x+3)y' + 2y = 0, \quad x > 0,$$

care are integralele particulare $y_1(x) = x^2 + 4x + 6$ şi $y_0(x) = (x-3)e^x$. Cu aceeaşi formulă (13) găsim

$$\int \frac{x^3 e^x}{(x^2 + 4x + 6)^2} dx = -C_0 + \frac{(x-3)e^x}{x^2 + 4x + 6}. \tag{18}$$

6. Metoda expusă permite atît regăsirea pe o cale mai simplă a unor rezultate cunoscute, cît mai ales obţinerea altor rezultate interesante.

Menţionăm faptul că prezintă mai mult interes exemplele în care se consideră diferite familii de funcţii ce reprezintă soluţiile generale ale unor ecuaţii diferenţiale care se construiesc ulterior, decît cazul ecuaţiilor diferenţiale date împreună cu soluţiile lor generale sau particulare.

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OBSERVATIONS SUR LE PROBLÈME DES CUADRATURES

(Résumé)

On développe une idée de G. Karapandjitch [2], [3] qui permet à calculer certaines primitives, en utilisant les solutions particulières de certaines équations différentielles simples.

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DIFFERENTIAL GEOMETRIC PROOF OF THE THEOREM ON THE CONSTRUCTION OF NOMOGRAMS IN R^3

BY

KATUHIKO MORITA and KENJI IKEDA

1. Introduction

In a previous paper [1], one of the authors stated the theorems concerning the negative result of the possibility of constructing the three variable alignment charts in the 3-dimensional Euclidean space R^3, and gave a point-set theoretic proofs of the theorems.

In the present article we shall give a differential geometric proof of the main theorem.

2. The Main Theorem

Theorem. *Let $C_i (i = 1, 2, 3)$ be three mutually disjoint space curves of class C^2 in the 3-dimensional Euclidean space R^3. If any straight line passing through two points $P_i \in C_i$, $P_j \in C_j$ $(i \neq i\,;\ i, j = 1, 2, 3)$ always intersects with the third curve C_k $(k \neq i, j)$, then the curves $C_i (i = 1, 2, 3)$ are plane curves, which are contained in the same plane.*

Proof. When we write the equations of the given three space curves $C_i\ (i = 1, 2, 3)$ in R^3 in vector form

$$\mathbf{x}_i = \mathbf{x}_i(s_i), \qquad (i = 1, 2, 3), \tag{1}$$

where parameters s_i represent the curve lengths of the curves C_i, respectively.

From the hypothesis of the theorem which states that any straight line passing through two points P_1 and P_2 always intersects with the curve C_3 (Fig. 1), we obtain the following equation :

(2) $$\mathbf{x}_3 - \mathbf{x}_2 = \mu(\mathbf{x}_2 - \mathbf{x}_1),$$

or

(2′) $$\mu\mathbf{x}_1 - (\mu + 1)\mathbf{x}_2 + \mathbf{x}_3 = 0.$$

By the geometric meaning of μ, we have

(3) $$\mu = \mu(s_1, s_2),$$

where we assume that the function μ is of class C^2; and by hypothesis we can put

(4) $$\mu \neq 0, \quad -1, \quad \infty.$$

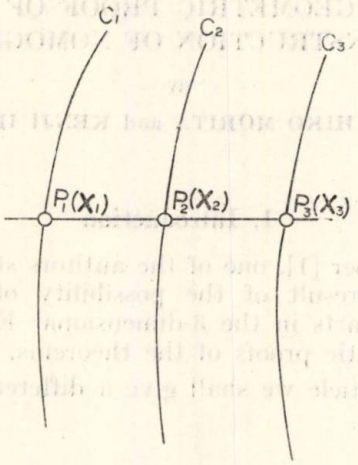

Fig. 1

Noting that s_3 is a function of s_1 and s_2 in (2′), that is,

(5) $$s_3 = f(s_1, s_2),$$

and assuming that the function f is also of class C^2, we differentiate (2′) with respect to s_1, s_2, respectively:

(6) $$\frac{\partial\mu}{\partial s_1}(\mathbf{x}_1 - \mathbf{x}_2) + \mu\mathbf{x}_1' + \frac{\partial f}{\partial s_1}\mathbf{x}_3' = 0,$$

(7) $$\frac{\partial\mu}{\partial s_2}(\mathbf{x}_1 - \mathbf{x}_2) - (\mu + 1)\mathbf{x}_2' + \frac{\partial f}{\partial s_2}\mathbf{x}_3' = 0,$$

where $\mathbf{x}_i' = \dfrac{d\mathbf{x}_i}{ds_i}$, $(i = 1, 2, 3)$.

Eliminating x_3' between (6) and (7), we have

$$p(s_1, s_2)(\mathbf{x}_1 - \mathbf{x}_2) + q(s_1, s_2)\mathbf{x}_1' + r(s_1, s_2)\mathbf{x}_2' = 0, \tag{8}$$

where

$$\begin{cases} p(s_1, s_2) \equiv \dfrac{\partial \mu}{\partial s_1}\dfrac{\partial f}{\partial s_2} - \dfrac{\partial \mu}{\partial s_2}\dfrac{\partial f}{\partial s_1}, \\ q(s_1, s_2) \equiv \mu \dfrac{\partial f}{\partial s_2}, \quad r(s_1, s_2) \equiv (\mu + 1)\dfrac{\partial f}{\partial s_1}. \end{cases} \tag{9}$$

As a functional relation does not exist between μ and s^{ε}, Jacobian

$$J\left(\frac{\mu, s_3}{s_1, s_2}\right) = J\left(\frac{\mu, f}{s_1, s_2}\right) = \begin{vmatrix} \dfrac{\partial \mu}{\partial s_1} & \dfrac{\partial f}{\partial s_1} \\ \dfrac{\partial \mu}{\partial s_2} & \dfrac{\partial f}{\partial s_2} \end{vmatrix} \neq 0 .$$

Hence from (9), we have

$$p(s_1, s_2) \neq 0. \tag{10}$$

Next, since $\mu \neq 0$, $\mu + 1 \neq 0$, $\dfrac{\partial f}{\partial s_1} \neq 0$ and $\dfrac{\partial f}{\partial s_2} \neq 0$, we have also from (9)

$$q(s_1, s_2) \neq 0, \qquad r(s_1, s_2) \neq 0. \tag{11}$$

Now differentiating (8) with respect to s_1, we have

$$\frac{\partial p}{\partial s_1}(\mathbf{x}_1 - \mathbf{x}_2) + \left(p + \frac{\partial q}{\partial s_1}\right)\mathbf{x}_1' + q\mathbf{x}_1'' + \frac{\partial r}{\partial s_1}\mathbf{x}_2' = 0 . \tag{12}$$

Eliminating x_2' between (8) and (12), we obtain

$$A(s_1, s_2)(\mathbf{x}_1 - \mathbf{x}_2) + B(s_1, s_2)\mathbf{x}_1' + C(s_1, s_2)\mathbf{x}_1'' = 0, \tag{13}$$

where

$$\begin{cases} A(s_1, s_2) \equiv p\dfrac{\partial r}{\partial s_1} - r\dfrac{\partial p}{\partial s_1}, \quad B(s_1, s_2) \equiv q\dfrac{\partial r}{\partial s_1} - r\dfrac{\partial q}{\partial s_1} - rp, \\ C(s_1, s_2) = -qr. \end{cases} \tag{14}$$

Now we consider (13) as a system of linear algebraic equations with unknowns A, B and C, we have a following equation as a necessary and sufficient condition for the nontrivial solution (cf. Remark 1 below).

$$|\, \mathbf{x}_1(s_1) - \mathbf{x}_2(s_2) \quad \mathbf{x}_1'(s_1) \quad \mathbf{x}_1''(s_1) \,| = 0. \tag{15}$$

Now the following equation

$$(16) \qquad | \mathbf{X} - \mathbf{x}_1(s_1) \quad \mathbf{x}_1'(s_1) \quad \mathbf{x}_1'(s_1) | = 0$$

is an equation of an osculating plane passing through any point s_1 on the curve C_1, and (15) is nothing but a condition that the osculating plane (16) passes through a point $\mathbf{x}_2(s_2)$ on the curve C_2. Hence, by the well-known theorem in the classical differential geometry, space curve C_1 is a plane curve on a plane Π, that is, $C_1 \subset \Pi$.

Furthermore, we can say that $\forall \mathbf{x}_2(s_2) \in \Pi$ from the above discussion ; hence $C_2 \subset \Pi$.

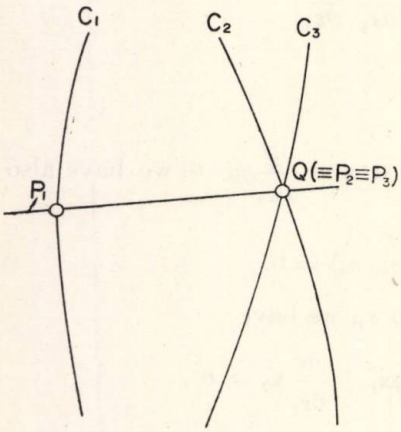

Fig. 2

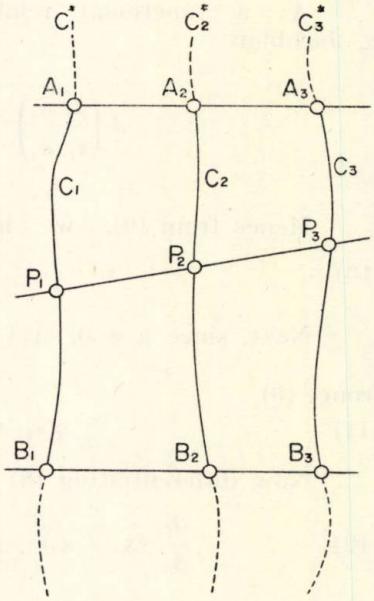

Fig. 3

As by similar line of the above argument it is easily proved that $C_3 \subset \Pi$, we have finally

$$C_1 \cup C_2 \cup C_3 \subset \Pi.$$

Thus we can complete the proof.

Remark 1. It is easily shown in (14) that $A(s_1, s_2) \neq 0$, $C(s_1, s_2) \neq 0$.

Remark 2. In the above proof we assumed that $\mu \neq 0$, $\mu \neq -1$ and $\mu \neq \infty$ by (4). When $\mu = 0$, for example, the curve C_2 intersects with C_3 at a point Q (Fig. 2). Points of neighborhoods of Q ($Q \in C_1$, C_2) being all contained in Π, point Q must also be on Π by the continuity of C_2 and C_3.

Similar results also hold for $\mu = -1$ and $\mu = \infty$.

Hence the main theorem holds when $\mu = 0, -1$ and ∞.

Remark 3. For the three space curves C_i of class C^2, of which lengths are finite, respectively, the main theorem also holds.

To verify this fact, we consider the osculating planes Π_i and Π'_i at the points A_i and B_i of the curves C_i $(i = 1, 2, 3)$, respectively (Fig. 3), and elongating adequately the curves C_i to both directions on these osculating planes, we have the curves of infinite lengths $C_i^* \supset C_i$ $(i = 1, 2, 3)$.

Hence by the hypothesis of the main theorem, it is easily shown that $\Pi_i \equiv \Pi'_i \equiv \Pi$, and therefore, $C_i \subset \Pi$ $(i = 1, 2, 3)$.

Remark 4. Similar proofs shows that the main theorem and Remark 2 and 3 hold, respectively, even when two space curves C_j, C_k $(j \neq k)$ become the same curve; therefore, this Remark 4 may be applied to the special type of Soreau-type chart and Clark-type one.

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DEMONSTRAREA CU MIJLOACELE GEOMETRIEI DIFERENŢIALE A TEOREMEI PRIVIND CONSTRUCŢIA NOMOGRAMELOR ÎN R^3

(Rezumat)

Se dă o nouă demonstraţie, care utilizează mijloace ale geometriei diferenţiale a curbelor, pentru teorema principală din [1], privind imposibilitatea construcţiei în spaţiul euclidean tridimensional a nomogramelor cu puncte aliniate, în trei variabile.

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LA MÉTHODE DES COORDONNÉES VECTORIELLES PLANE-AXIELLES DANS LA DÉTERMINATION DES VITESSES DANS LE MOUVEMENT GÉNÉRAL DU SOLIDE RIGIDE

PAR

C. HUIU, ELISABETA RUSU et N IRIMICIUC

0. Introduction

Tout en poursuivant l'idée développée dans [6], les auteurs de cette Note utilisent la méthode des coordonnées vectorielles plane-axielles, exposée dans la suite de travaux [2]...[5], pour la détermination des distributions des vitesses des solides rigides en mouvement spaciel.

On adopte toujours les mêmes conventions de notation comme dans [6].

1. Composantes plane-axielles des paramètres cinématiques de premier ordre du mouvement général du solide

Les composantes plane-axielles des paramètres cinématiques $\bar{v}_Q$ et $\bar{\omega}$ du solide rigide en mouvement spaciel sont déterminées, à l'instant t quelquonque, par les relations suivantes :

$$(1.1)\quad \begin{cases} \bar{v}_{Q_\Pi} = \sigma_v \bar{v}^r_{Q_\Pi} = \sigma_v[\dot{x}^r_{Q_1}\bar{i}_1 + \dot{x}^r_{Q_2}\bar{i}_2] = \sigma_v[\dot{r}^r_Q \bar{i}_{r_Q} + r^r_Q\dot{\varphi}^r_Q \bar{i}_{\varphi_Q}], \\ \bar{v}_{Q_\Delta} = \sigma_v \bar{v}^r_{Q_\Delta} = \sigma_v[\dot{x}^r_{Q_3}\bar{i}_3] = \sigma_v[\dot{z}^r_Q \bar{i}_{z_Q}], \end{cases}$$

et :

$$(1.2)\quad \begin{cases} \bar{\omega}_\Pi = \sigma_\omega \bar{\omega}^r_\Pi = \sigma_\omega[\dot{\theta}^r_n \bar{n}_1 - \dot{\theta}^r_r \sin\theta_n \bar{n}_2], \\ \bar{\omega}_\Delta = \sigma_\omega \bar{\omega}^r_\Delta = \sigma_\omega[\dot{\theta}^r_p + \dot{\theta}^r_r \cos\theta_n]\bar{u}_\Delta, \end{cases}$$

où :

(1.3) $$\sigma_\omega = \frac{\nu_\omega}{q_\omega} \quad \left(\frac{1/\text{sec}}{\text{cm}}\right)$$

est l'échelle de représentation de la vitesse angulaire et

(1.4) $$\sigma_v = \sigma_L \sigma_\omega \,.\, 1\ \text{cm} = \frac{\nu_L}{q_L}\frac{\nu_\omega}{q_\omega} \quad \left(\frac{\text{m/s}}{\text{cm}}\right)$$

est l'échelle de représentation des vitesse linéaires.

Dans les relations (1.1) et (1.2) les lettres affectées de l'indice supérieur (^{r}) désignent les segments représentatifs des valeurs à l'instant t, des paramètres de position du solide, respectivement de leurs dérivées par rapport au temps.

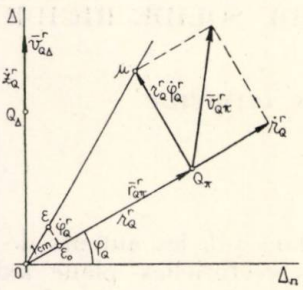

Fig. 1

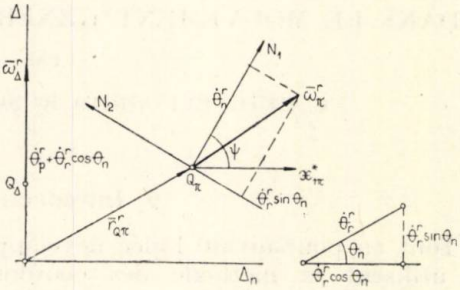

Fig. 2

Les constructions graphiques pour la détermination des vecteurs représentatifs $\bar{v}^r_{Q_\Pi}$ et $\bar{v}^r_{Q_\Delta}$ sont indiquées dans la fig. 1 et celles qui regardent la détermination des vecteurs représentatifs $\bar{\omega}^r_\Pi$ et $\bar{\omega}^r_\Delta$, dans la fig. 2.

2. Détermination de la vitesse d'un point arbitraire du solide en mouvement général

Les procédées graphiques élaborés pour la détermination des distributions des vitesses sont fondés sur les deux observations suivantes :

a) Tous les points du solide situés sur un parallèle à la ligne d'action du vecteur $\bar{\omega}$ ont la même vitesse, c'est-à-dire qu'on peut écrire (fig. 3) :

(2.1) $$\bar{v}_{B_1} = \bar{v}_Q + \bar{\omega} \times \bar{r}'_{B_1} = \bar{v}_Q + \bar{\omega} \times \bar{r}'_B = \bar{v}_B \,.$$

Comme conséquence, la vitesse du point arbitraire B du solide, (fig. 3), pourra-t-être déterminée à l'aide du point B_0, d'intersection entre le plan $Qx_1^*x_2^*$ — parallèle au plan de la représentation et le parallèle mené par B à la ligne d'action du vecteur $\bar{\omega}$.

b) La vitesse de rotation du point arbitraire B est égale au moment par rapport au pôle Q, invariablement lié au solide, du vecteur $\bar{\omega}$, considéré appliqué en B (noté $\bar{\omega}_B$), c'est-à-dire qu'on peut écrire :

$$\bar{v}_{\mathrm{rot}_B} = \bar{\omega} \times \bar{r}'_B = -\overline{\mathcal{M}}_Q(\bar{\omega}_B) \tag{2.2}$$

et donc :

$$\bar{v}_B = \bar{v}_Q - \overline{\mathcal{M}}_Q(\bar{\omega}_B)\,. \tag{2.3}$$

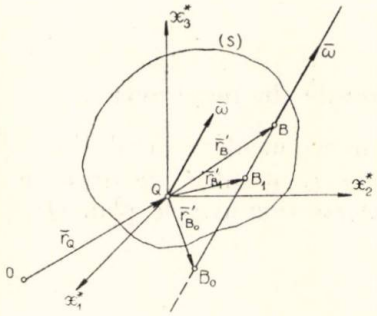

Fig. 3

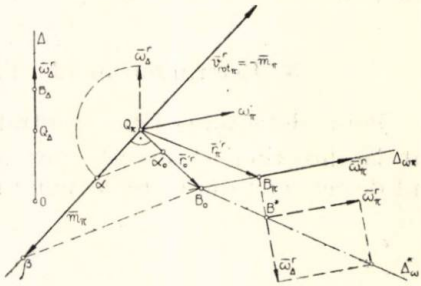

Fig. 4

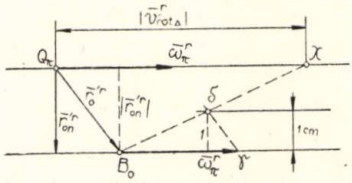

Fig. 5

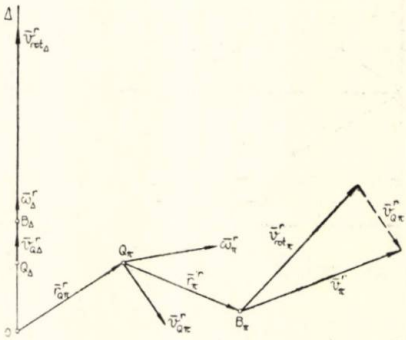

Fig. 6

Les composantes plane-axielles de la vitesse $\bar{v}_B = \bar{v}_{B_0}$ seront données par les relations :

$$\begin{cases} \bar{v}_{B_\Pi} = \sigma_v \bar{v}^r_\Pi = \sigma_v(\bar{v}^r_{Q_\Pi} - \bar{m}_\Pi), \\ \bar{v}_{B_\Delta} = \sigma_v \bar{v}^r_\Delta = \sigma_v(\bar{v}^r_{Q_\Delta} + \bar{v}^r_{\mathrm{rot}_\Delta}), \end{cases} \tag{2.4}$$

où $\bar{m}_\Pi$ désigne le vecteur représentatif de la composante plane du vecteur $\overline{\mathcal{M}}_Q(\bar{\omega}_B)$, déterminé à l'aide des constructions graphiques indiquées

dans la fig. 4 et $\bar{v}^r_{\mathrm{rot}_\Delta}$ est la composante axielle du vecteur représentatif de la vitesse de rotation du point B_C, ayant le module déterminé graphiquement dans la fig. 5.

Finalement, la fig. 6 indique les constructions graphiques qui donnent les composantes plane-axielles $\bar{v}^r_\Pi$ et $\bar{v}^r_\Delta$ du vecteur représentatif

$$\bar{v}^r_B = \bar{v}^r_\Pi + \bar{v}^r_\Delta \tag{2.5}$$

de la vitesse du point B.

3. Détermination de l'axe centrale du mouvement

Pour déterminer l'axe centrale du mouvement général du solide, dont la direction est donnée par le vecteur $\bar{\omega}$, il suffit de fixer un seul point de cet axe, qui serra le point C_0 d'intersection avec le plan $Qx_1^*x_2^*$.

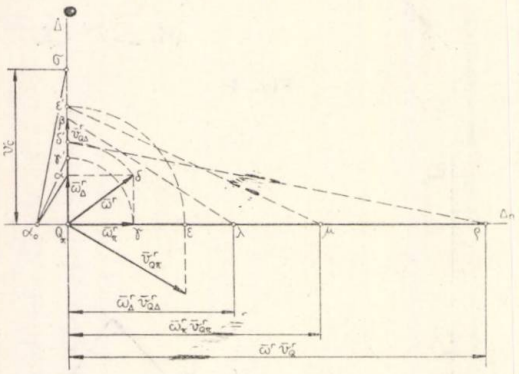

Fig. 7

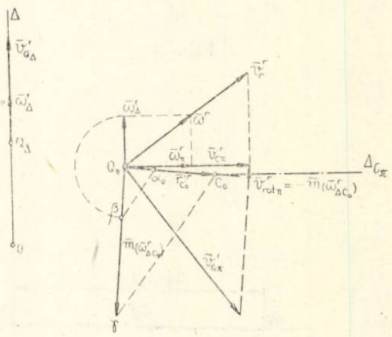

Fig. 8

La valeur scalaire du vecteur représentatif de la vitesse de translation minime — donc la vitesse du point C_0 — donnée par la relation

$$v^r_C = \frac{1}{|\bar{\omega}^r|}(\bar{v}^r_{Q_\Delta}\bar{\omega}^r_\Delta + \bar{v}^r_{Q_\Pi}\bar{\omega}^r_\Pi), \tag{3.1}$$

sera déterminée à l'aide des constructions graphiques indiquées dans la fig. 7.

Par la projection du vecteur $\bar{v}^r_C$ sur la ligne d'action de la composante $\bar{\omega}^r_\Pi$, on obtient la composante plane $\bar{v}^r_{C_\Pi}$, qui permet ensuite de déterminer le vecteur :

$$\overline{m}_{\Pi}(\overline{\omega}^{r}_{\Delta_{C_0}}) = \overline{v}^{r}_{Q_{\Pi}} - \overline{v}^{r}_{C_{\Pi}} = -\overline{v}^{r}_{\mathrm{rot}\Pi}, \tag{3.2}$$

représentant le moment par rapport au pôle Q du vecteur $\overline{\omega}^{r}_{\Delta}$, considéré appliqué en C_0 et qui est obtenu à l'aide des constructions graphiques indiquées dans la fig. 8.

L'égalité évidente :

$$\overline{m}_{\Pi}(\overline{\omega}^{r}_{\Delta_{C_0}}) \cdot 1\ \mathrm{cm} = \overline{r}'^{r}_{C_0} \times \overline{\omega}^{r}_{\Delta_{C_0}} \tag{3.3}$$

permet aussi de déterminer le vecteur $\overline{r}'^{r}_{C_0}$, (fig. 8), qui donnera la position de point C_0.

Il suffit maintenant de mener par C_0 un parallèle à la ligne d'action du vecteur $\overline{\omega}^{r}_{\Pi}$, pour obtenir ainsi la projection, sur le plan de représentation, de l'axe centrale du mouvement général du solide.

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Institut Polytechnique, Jassy,
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et
Chaire de Mécanique théorique

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METODA COORDONATELOR VECTORIALE PLAN-AXIALE ÎN DETERMINAREA VITEZELOR ÎN MIŞCAREA GENERALĂ A SOLIDULUI RIGID

(Rezumat)

Urmărindu-se, în continuare, ideea expusă în lucrarea [6], de a se crea, cu ajutorul elementelor de geometrie reprezentativă plan-axială prezentate în lucrările [2]... ...[6], [9] şi [11], o nouă metodă grafică de rezolvare a problemelor spaţiale de mecanică, denumită de autori *metoda coordonatelor vectorilor plan-axiale*, în nota de faţă sînt indicate procedeele grafice de determinare a principalelor elemente caracterizînd distribuţia vitezelor în mişcarea spaţială a solidului şi anume : parametrii cinematici de ordinul întîi, viteza unui punct arbitrar al solidului şi axa centrală a mişcării generale a solidului.

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C. Z. 5158.

CONTRIBUŢII LA TRASAREA LINIILOR DE INTERSECŢIE A DOUĂ CORPURI CARE AU BAZELE ÎN PLANE DIFERITE, DE POZIŢIE GENERALĂ

DE

AL. MATEI, VICTOR GABA şi TATIANA TACU

1. Introducere

În lucrarea de faţă se prezintă o metodă grafică simplă şi eficientă pentru determinarea vizibilităţii liniei sau liniilor de intersecţie a două corpuri care au bazele în plane diferite, de poziţie generală.

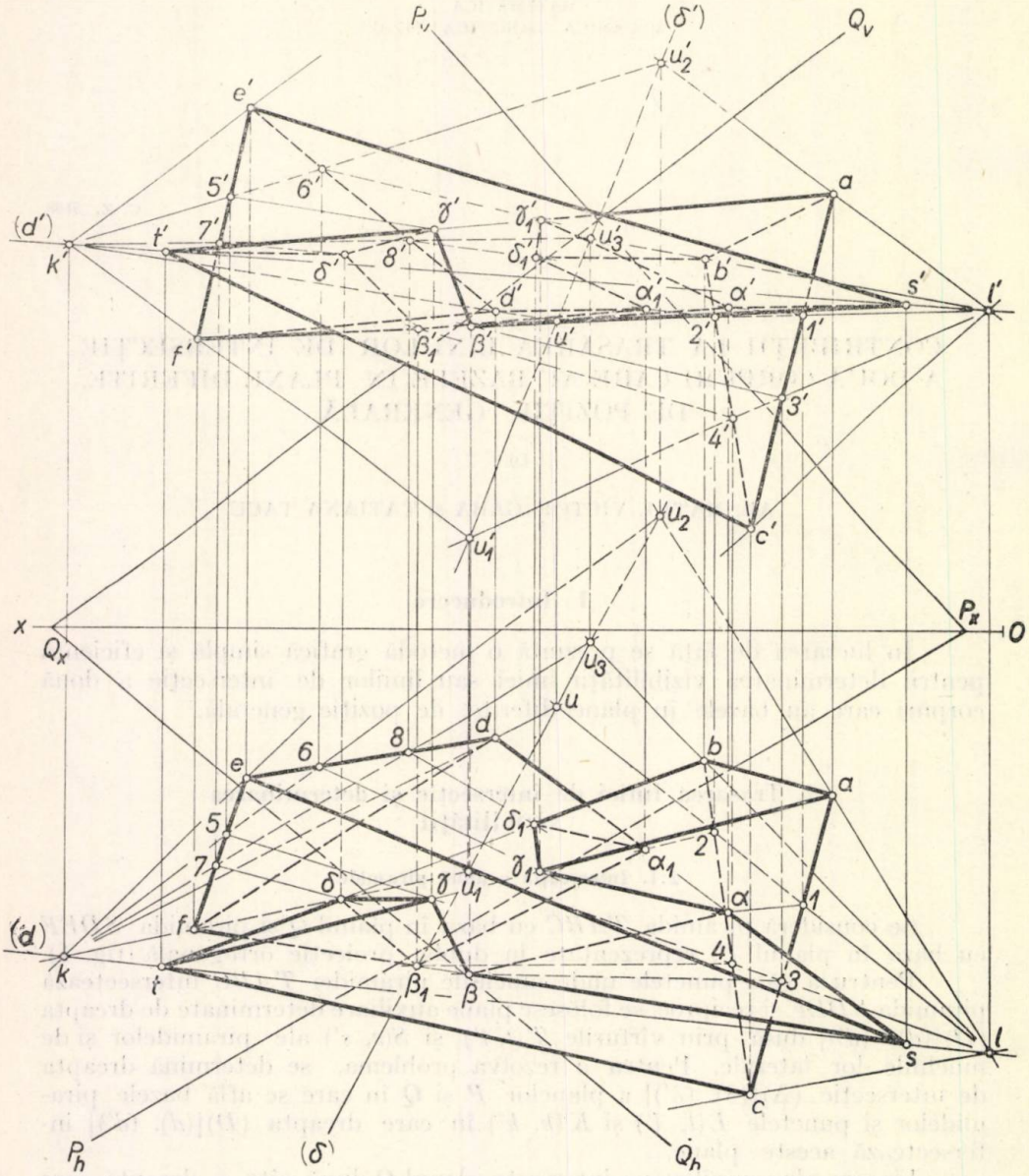
2. Trasarea liniei de intersecţie şi determinarea vizibilităţii

2.1. Intersecţia a două piramide

Se consideră piramida $TABC$ cu baza în planul Q şi piramida $SDEF$ cu baza în planul P, reprezentate în dublă proiecţie ortogonală (fig. 1).

Pentru a găsi punctele unde muchiile piramidei $TABC$ intersectează piramida $SDEF$ şi reciproc, se folosesc plane auxiliare determinate de dreapta $(D)[(d), (d')]$ dusă prin vîrfurile $T(t, t')$ şi $S(s, s')$ ale piramidelor şi de muchiile lor laterale. Pentru a rezolva problema, se determină dreapta de intersecţie $(\Delta)[(\delta), (\delta')]$ a planelor P şi Q în care se află bazele piramidelor şi punctele $L(l, l')$ şi $K(k, k')$ în care dreapta $(D)[(d), (d')]$ intersectează aceste plane.

Fiecare plan auxiliar va intersecta planul Q după cîte o dreaptă care trece prin punctul $L(l, l')$ şi planul P după cîte o dreaptă care trece prin punctul $K(k, k')$. Cele două drepte se întîlnesc într-un punct situat pe dreapta $(\Delta)[(\delta), (\delta')]$.


P_v
(δ')
Q_v
u_2'
e'
$5'$
$6'$
a
(d')
t'
$7'$
γ'
γ_1'
u_3'
k'
$8'$
δ_1'
b'
δ'
d'
α_1'
α'
s'
l'
f'
β_1'
β'
u'
$2'$
$1'$
$3'$
$4'$
u_2
c'
u_1'
x
Q_x
P_x
O
u_3
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d
8
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b
e
a
5
δ_1
2
7
γ_1
α_1
δ
γ
u_1
α
1
(d)
f
k
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4
3
β_1
β
s
l
c
P_h
(δ)
Q_h

Fig. 1

Se consideră planul auxiliar prin muchia SD. Acesta intersectează planul P după dreapta $KD(kd, k'd')$ care determină punctul $U(u, u')$ pe dreapta (Δ) $[(\delta), (\delta')]$. Cunoscînd punctul U (u, u') se poate duce şi dreapta de intersecţie dintre planul auxiliar considerat şi planul Q, ştiind că această dreaptă trece prin punctele $U(u, u')$ şi L (l, l').

Dreapta UL $(ul, u'l')$, intersectează poligonul de bază al piramidei $TABC$ în punctele $I(1, 1')$ şi $II(2, 2')$. Cunoscînd punctele I şi II se determină triunghiul $T-I-II$ $(t-1-2, t'-1'-2')$ după care planul auxiliar secţionează piramida $TABC$. Laturile triunghiului sînt intersectate de muchia SD în punctele (α, α') şi (α_1, α_1').

În continuare se consideră planul auxiliar prin muchia SF, care este unul din planele limită. Acest plan intersectează planul P după dreapta KF $(kf, k'f')$ care determină pe (Δ) $[(\delta), (\delta')]$ punctul U_1 (u_1, u_1'). Cunoscînd acest punct se poate construi şi dreapta U_1L $(u_1l, u_1'l')$, ca intersecţie dintre planul auxiliar (ce trece prin muchia SF) şi planul Q.

Această dreaptă intersectează poligonul de bază al piramidei $TABC$ în punctele III $(3, 3')$ şi IV $(4, 4')$. Laturile triunghiului $T-III-IV$ $(t-3-4, t'-3'-4')$ după care planul auxiliar secţionează piramida $TABC$, intersectează muchia SF în punctele (β, β') şi (β_1, β_1').

Planul auxiliar determinat de muchia TA este al doilea plan limită. El intersectează planul Q după dreapta LA $(la, l'a')$ şi determină punctul U_2 (u_2, u_2'). Cunoscînd acest punct se duce dreapta U_2K $(u_2k, u_2'k')$, de intersecţie dintre planul auxiliar (ce trece prin muchia TA) şi planul P. Această dreaptă intersectează poligonul de bază al piramidei $SDEF$ în punctele V $(5, 5')$ şi VI $(6, 6')$. Laturile triunghiului $S-V-VI$ $(s-5-6, s'-5'-6')$ după care planul auxiliar secţionează piramida $SDEF$, intersectează muchia TA în punctele (γ, γ') şi (γ_1, γ_1').

Procedînd în mod analog cu planul auxiliar dus prin muchia TB, se determină (δ, δ') şi (δ_1, δ_1').

Planele limită indică că genul intersecţiei este o rupere.

Pentru a stabili ordinea de unire a punctelor se foloseşte metoda mobilului, proiectat pe planele P şi Q în care se găsesc bazele piramidelor. Se întocmeşte tabelul 1 trecînd în rîndul 1 proiecţiile mobilului pe planul Q (baza piramidei $TABC$), în rîndul 2 proiecţiile mobilului pe planul P (baza piramidei $SDEF$) şi în rîndul 3, ordinea de unire a punctelor.

Se admite că în momentul iniţial, mobilul se găseşte în planul auxiliar determinat de muchia TA şi dreapta (D) $[(d), (d')]$, proiecţiile mobilului pe cele două plane fiind : a pe planul Q şi 5 pe planul P. Deplasînd mobilul pînă ce proiecţia lui pe planul Q ajunge în b şi proiecţia pe planul P în 7, el ajunge în planul auxiliar determinat de muchia TB şi dreapta (D) $[(d), (d')]$.

Urmărind în continuare deplasarea proiecţiilor mobilului pe cele două baze, se stabileşte ordinea de unire a punctelor, trecută în rîndul 3 al tabelului 1.

Tabelul 1

Piramida *TABC*	*a*	*b*	*4*	*2*	*b*	*a*	*1*	*3*	*a*
Piramida *SDEF*	*5*	*7*	*f*	*d*	*8*	*6*	*d*	*f*	*5*
Ordinea de unire a punctelor	γ	δ	β_1	α_1	δ_1	γ_1	α	β	γ
Vizibilitatea punctelor în proiecţie orizontală	*v*	*v*	*n*	*n*	*v*	*v*	*v*	*v*	*v*
Vizibilitatea punctelor în proiecţie verticală	*v*	*n*	*n*	*n*	*n*	*n*	*n*	*v*	*v*

Vizibilitatea poligonului de intersecţie se stabileşte pe principiul că *dacă două puncte sînt văzute, linia care le uneşte este văzută, iar dacă unul sau ambele puncte sînt nevăzute, linia care le uneşte este nevăzută.* Această regulă are următoarea excepţie : dacă în şirul de puncte văzute, care reprezintă vîrfurile poligonului de intersecţie, se găsesc două puncte succesive care aparţin conturului aparent al aceluiaşi poliedru, latura determinată de cele două puncte este nevăzută.

Urmărind proiecţiile orizontale ale punctelor *A* şi *V* se observă că acestea sînt văzute, deci muchia *TA* şi dreapta *SV* sînt văzute ; înseamnă că în această proiecţie punctul lor de intersecţie γ este văzut şi se notează deci *v* în tabelul 1 pe coloana *a*, *5*, γ, rîndul *4*.

Punctul următor δ este văzut deoarece muchia *TB* şi dreapta *S—VII* care îl determină, sînt văzute. Punctul β_1 nu este văzut, deoarece în această proiecţie dreapta *T—IV* nu este văzută. De asemenea nici punctul α_1 nu este văzut, deoarece dreapta *T—II* nu este văzută.

În continuare punctele δ_1, γ_1, α, β şi γ sînt văzute. Deoarece în şirul acesta de puncte văzute există două puncte succesive α şi β ce aparţin conturului aparent al piramidei *SDEF*, se aplică excepţia la regula vizibilităţii menţionată anterior. Deci, cu toate că aceste două puncte sînt văzute, linia ce le uneşte este nevăzută şi se trasează cu linie întreruptă.

Urmărind proiecţiile verticale ale punctelor *A* şi *V* se observă că ele sînt văzute ; înseamnă că muchia *TA* şi dreapta *S—V* sînt de asemenea văzute, deci şi punctul lor de intersecţie γ' este văzut. Punctul următor δ' este nevăzut, deoarece în această proiecţie muchia *TB* este nevăzută. În continuare punctele β_1', α_1', δ_1', γ_1' şi α' sînt nevăzute iar punctele β' şi γ' sînt văzute, fapt consemnat în tabelul 1.

Avînd rîndul 5 din tabelul 1 cu vizibilitatea în proiecţie verticală completat se urmăreşte dacă în şirul de puncte văzute există două puncte succesive care să aparţină conturului aparent al aceluiaşi poliedru. În situaţia de faţă nu se aplică excepţia menţionată.

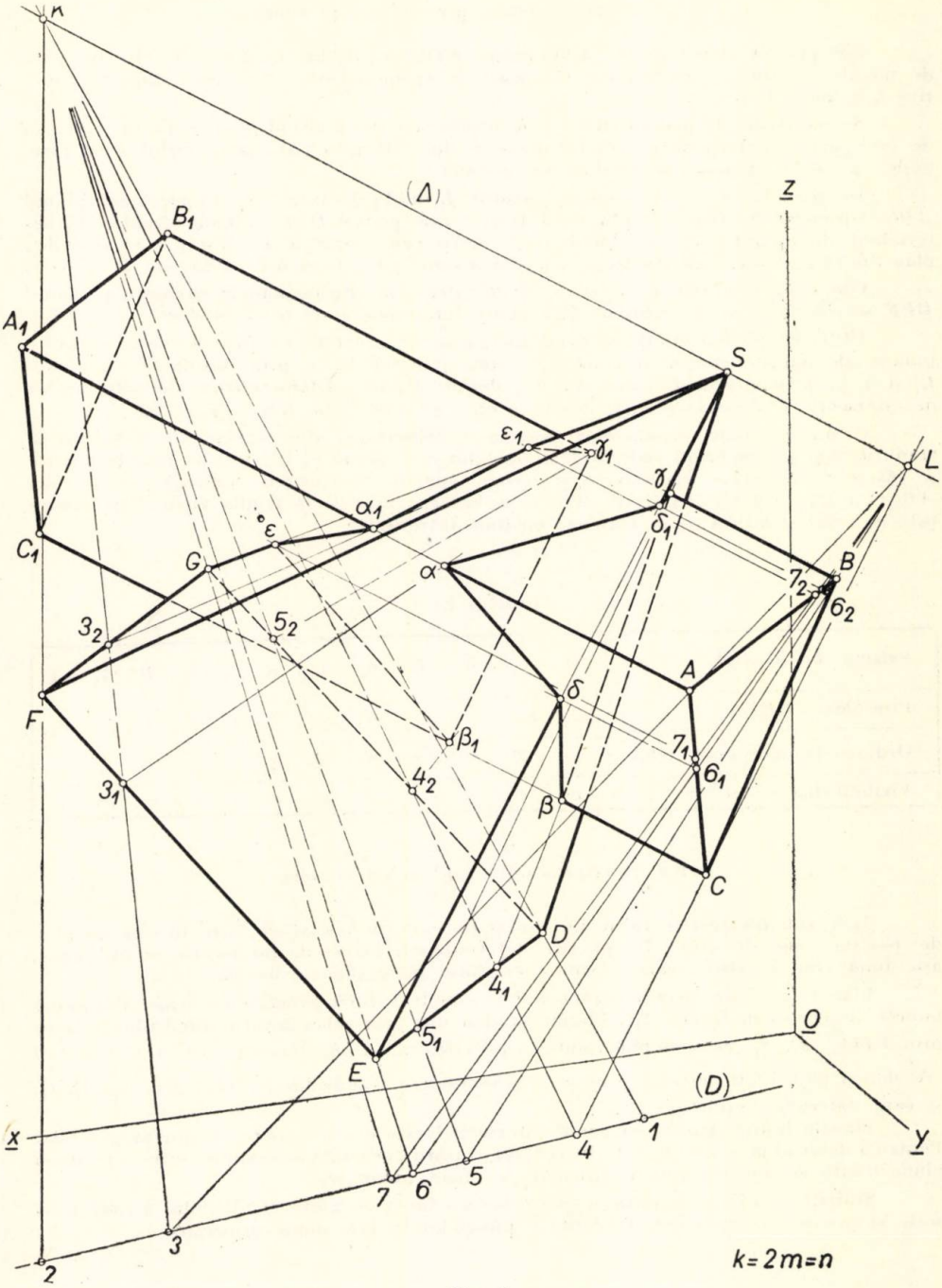
K
(Δ)
Z
B1
A1
S
ε1
γ1
L
γ
α1
ε
δ1
C1
G
α
B
72
62
32
52
A
δ
F
β1
71
61
42
31
β
C
D
41
O
51
E
(D)
x
y
1
4
5
6
7
3
2
k=2m=n

Fig. 2

2.2. Intersecţia unei prisme cu o piramidă

Fie prisma $ABCC_1A_1B_1$ şi piramida $SDEFG$, fiecare cu baza în cîte un plan de poziţie generală, reprezentate în proiecţia axonometrică ortogonală dimetrică mărită $k = 2m = n$ (fig. 2).

Se cercetează în primul rînd genul intersecţiei, considerînd planele limită. În cazul de faţă pentru determinarea acestor plane se duce dreapta (Δ) prin vîrful S al piramidei, paralelă cu muchiile laterale ale prismei.

În continuare se determină punctul L unde dreapta (Δ) intersectează planul ABC şi punctul K unde dreapta (Δ) intersectează planul DEF. Planul limită L_1 determinat de dreapta (Δ) şi muchia AA_1 trece prin punctele L, 3 şi K, iar al doilea plan limită L_2 determinat de dreapta (Δ) şi muchia CC_1, trece prin punctele L, 4 şi K.

Observînd dreptele $4-K$ şi $3-K$ de intersecţie dintre planele auxiliare şi planul DEF se conchide că în cazul de faţă genul intersecţiei este o pătrundere.

Utilizînd planele limită şi urmărind epura se constată că sînt de acum obţinute puncte ale poligoanelor de intersecţie, anume punctele α, α_1 prin intermediul planului L_1 şi β, β_1 prin intermediul planului L_2. Pentru a găsi celelalte vîrfuri ale poligoanelor de intersecţie se folosesc planele auxiliare duse prin muchiile BB_1, SE şi SG.

Ordinea de unire a punctelor, precum şi vizibilitatea, sînt consemnate în tabelul 2, rîndurile 3 şi 4. Urmărind vizibilitatea punctelor se constată că în şirul de puncte văzute se găsesc două puncte succesive β şi γ care aparţin conturului aparent al prismei $ABCC_1A_1B_1$. Deci în situaţia de faţă se aplică excepţia de la regula vizibilităţii enunţată anterior şi latura $\beta\gamma$ se trasează cu linie întreruptă.

Tabelul 2

Prisma $ABCC_1A_1B_1$	A	6_1	C	B	6_2	A	A	7_1	C	B	7_2	A
Piramida $SDEFG$	3_1	E	4_1	5_1	E	3_1	3_2	G	4_2	5_2	G	3_2
Ordinea de unire a punctelor	α	δ	β	γ	δ_1	α	α_1	ε	β_1	γ_1	ε_1	α_1
Vizibilitatea punctelor	v	v	v	v	v	v	v	v	n	n	v	v

2.3. Intersecţa unui con cu un cilindru

În cazul intersecţiei dintre un con şi un cilindru fiecare cu baza în cîte un plan de poziţie generală (fig. 3), pentru determinarea curbei de intersecţie se utilizează, aşa după cum se ştie, plane auxiliare cuprinse între planele limită.

Planele auxiliare secţionează conul şi cilindrul după generatoare, care determină puncte ale curbei de intersecţie. Planul auxiliar dus prin generatoarea cilindrului ce trece prin VIII_1 (8_1, $8_1'$) este un plan limită. Acesta determină punctele Λ (λ, λ') şi Λ_1 (λ_1, λ_1'). Al doilea plan limită este cel dus prin generatoarea cilindrului ce trece prin I_1 (1_1, $1_1'$) şi care determină punctul (α, α').

Planele limită denotă că genul intersecţiei este o pătrundere simplu tangenţială. Pentru a determina şi alte puncte în vederea trasării curbei de intersecţie, între cele două plane limită se duc un număr suficient de plane auxiliare.

Stabilirea ordinei de unire a punctelor s-a făcut cu ajutorul tabelului 3 care foloseşte şi pentru determinarea vizibilităţii punctelor în cele două proiecţii.

Tabelul 3

Conul	1_2	2_3	3_3	4_3	5_3	6_3	7_3	8_3	7_3	6_3	5_3	4_3	3_3	2_3	1_2	2_4	3_4	4_4	5_4	6_4	7_4	8_2	7_4	6_4	5_4	4_4	3_4	2_4	1_2
Cilindrul	1_1	2_2	3_2	4_2	5_2	6_2	7_2	8_1	7_1	6_1	5_1	4_1	3_1	2_1	1_1	2_2	3_2	4_2	5_2	6_2	7_2	8_1	7_1	6_1	5_1	4_1	3_1	2_1	1_1
Ordinea de unire a punctelor	α	β_1	γ_1	δ_1	ε_1	ζ_1	η_1	λ	η	ζ	ε	δ	γ	β	α	β_2	γ_2	δ_2	ε_2	ζ_2	η_2	λ_1	η_3	ζ_3	ε_3	δ_3	γ_3	β_3	α
Vizibilitatea punctelor în proiecţie orizontală	*v*	*v*	*v*	*v*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*v*	*v*	*v*	*v*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*v*	*v*
Vizibilitatea punctelor în proiecţie verticală	*v*	*v*	*v*	*v*	*n*	*n*	*n*	*n*	*n*	*n*	*n*	*v*	*v*	*v*	*v*	*v*	*v*	*v*	*v*	*n*	*n*	*n*	*n*	*n*	*v*	*v*	*v*	*v*	*v*

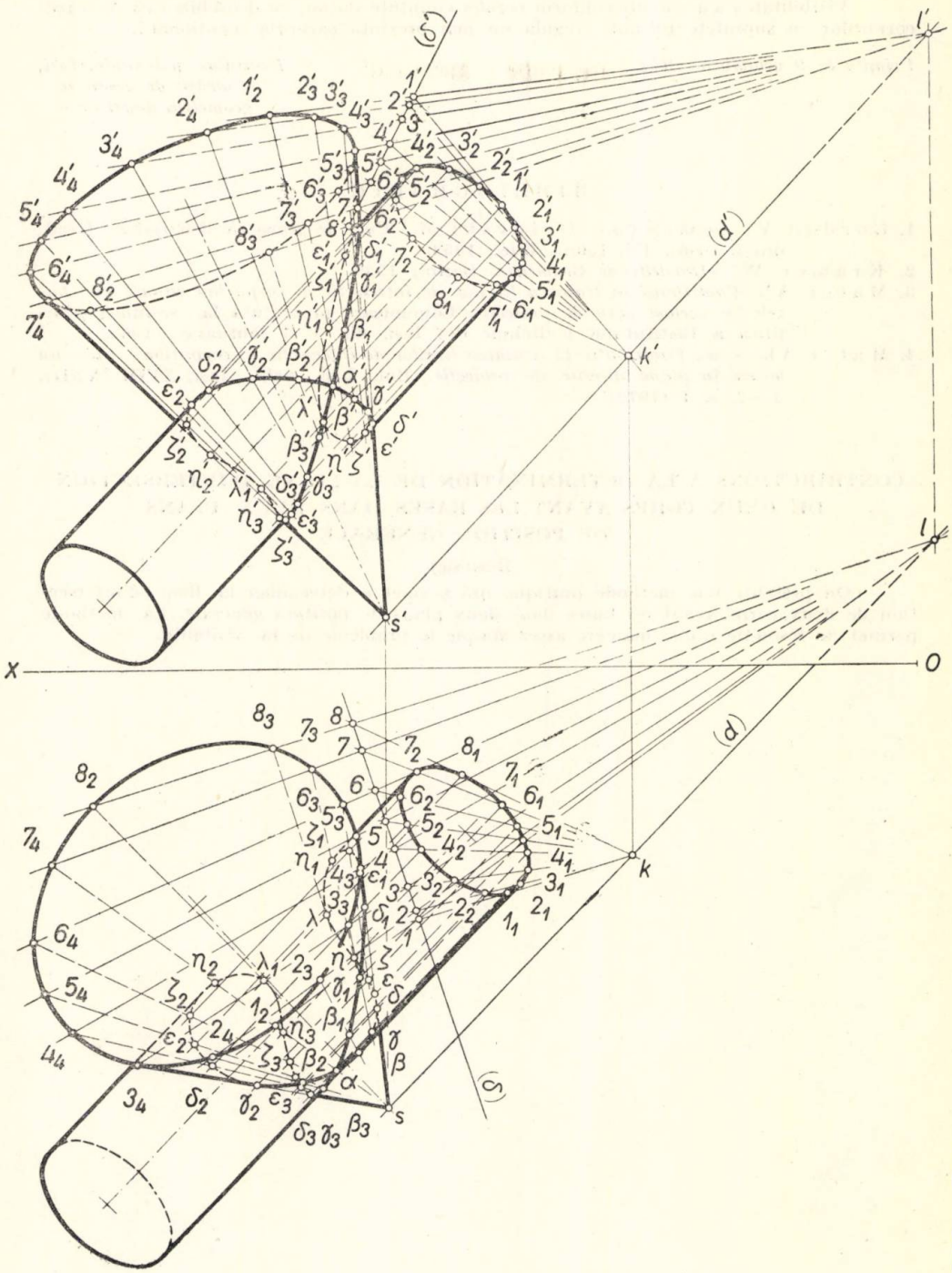

Fig. 3

Vizibilitatea s a stabilit conform regulei enunţate iniţial, cu deosebirea că în cazul corpurilor cu suprafeţe rotunde, regula nu mai prezintă excepţia menţionată.

Primită la 9 octombrie 1972

Institutul politehnic, Iaşi,
Catedra de desen şi
geometrie descriptivă

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CONTRIBUTIONS À LA DÉTERMINATION DE LA LIGNE D'INTERSECTION DE DEUX CORPS AYANT LES BASES DANS DEUX PLANS DE POSITION GÉNÉRALE

(Résumé)

On présente une méthode pratique qui permet à déterminer la ligne d'intersection de deux corps ayant les bases dans deux plans de position générale. La méthode permet de résoudre d'une manière assez simple le problème de la visibilité.

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Secţia I
MATEMATICĂ.
MECANICĂ TEORETICĂ. FIZICĂ

D C. 515.8

PROIECŢIA CENTROAFINĂ CU DIRECTIA DE PROIECŢIE GENERALĂ [1])

DE

I. RUSU

0. Introducere

În lucrări anteriare, autorul a stabilit corespondenţa afină între proiecţiile ortogonale ale aceluiaşi obiect pe planele Π şi P şi a reuşit să construiască proiecţia obiectului pe planul P, cea de pe planul Π fiind dată.

În lucrarea de faţă se consideră proiecţia plan-axială a obiectului dat, care apoi se proiectează după o direcţie oarecare pe planul P. Între cele două proiecţii se stabileşte aceeaşi corespondenţă afină, care conduce a construcţia proiecţiei de pe planul P.

1. Proiecţia centroafină a punctului

Se consideră în sistemul de referinţă plan-axial punctul $A\,(a, \overline{a})$ şi planul P $(P_0 \cdot P_z)$ (fig. 1).

Prin punctul dat A se duce dreapta $D_1(d_1, D_{10})$, ca proiectantă a acestuia pe planul P, determinată de punctele $A\,(a, \overline{a})$ şi $M\,(m, \overline{m})$. Din intersecţia acestei proiectante cu planul considerat rezultă punctul $A_1\,(a_1, \overline{a_1})$, proiecţia punctului $A(a, \overline{a})$ dat pe planul P. Astfel, punctele $A(a, \overline{a})$ şi $A_1(a_1, \overline{a_1})$ sînt corespondente afine, iar dreapta D_1 ce le conţine este direcţia de proiectare.

Pentru a se obţine în planul reprezentării proiecţia reală din planul P, s-a considerat că punctul A_1 din acest plan este rotit în planul Q_1 în jurul urmei U_1 pînă ajunge în planul reprezentării, deci pe urma Q_{10}, pentru că

$$Q_{10} = [Q_1] \cap [P], \text{ iar } [Q_1] \supset D_1 \text{ şi } [Q_1] \cap [P] = \Delta_1,$$

deci

$$Q_{10} \equiv d_1 \equiv \delta_1.$$

---

[1]) Prezentată la sesiunea ştiinţifică a Institutului politehnic din Iaşi, din 28—30 septembrie 1967.

2. Proiecţia centroafină a figurilor plane

Fie pătratul $ABCE$ situat în planul reprezentării.

Prin vîrful A $(a, \bar{a})$ se duce dreapta D_1 (d_1, D_{10}), oblică faţă de planul de proiecţie P (P_0, P_z) (fig. 2). Prin această dreaptă, care este direcţia de prioectare, determinată de punctele A $(a, \bar{a})$ şi $M(m, \bar{m})$, se consideră planul Q_1 perpendicular pe planul reprezentării. Acesta intersectează planul P după dreapta Δ_1 (δ_1, Δ_{10}), notată în rabatere pe planul Π cu Δ_{10}.

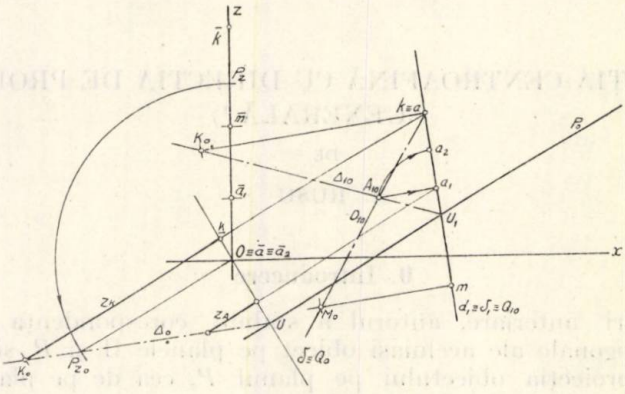

Fig. 1

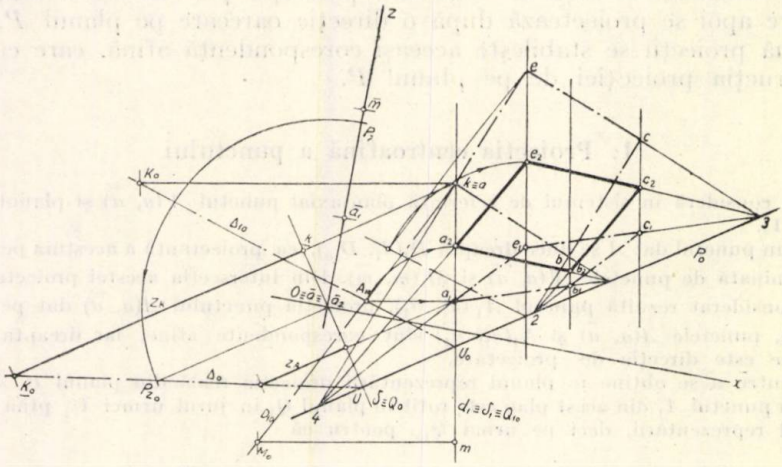

Fig. 2

Intersecţia dintre proiectanta D_1 (d_1, D_{10}) şi dreapta Δ_1 (δ_1, Δ_{10}) este punctul A_1 (a_1, A_{10}). În rabatere pe planul Π acest punct s-a notat cu A_{10}. Punctul A_1 fiind intersecţia proiectantei D_1 cu planul P, rezultă

că aceasta este proiecţia pe planul P a punctului considerat A, deci afinul său. Se observă că latura ab intersectează axa de afinitate în punctul 1. Unind printr-o dreaptă proiecţia a cu punctul 1, aceasta intersectează proiectanta vîrfului $B(b, \bar{b})$ în punctul b_1, care este afinul acestuia.

În mod analog s-a găsit în planul reprezentării şi proiecţiile c_1 şi e_1, care împreună cu a_1 şi b_1 determină proiecţia pătratului dat.

Cum paralelogramul $a_1b_1c_1e_1$ nu reprezintă proiecţia reală pe planul Π a pătratului dat, se rabate planul P cu proiecţia conţinută pe planul Π al reprezentării.

Aceasta se reduce la a trasa un arc de cerc cu centrul în U_0 şi de rază U_0A_{10}, care intersectează urma $Q_{10} \equiv d_1 \equiv \delta_1$ în punctul a_2. Acest punct permite construcţia paralelogramului $a_2b_2c_2e_2$, ca figură afină a pătratului dat şi care constituie mărimea reală a proiecţiei acestuia pe planul P.

3. Proiecţia oblică a unui cerc dat, situat în planul Π al reprezentării

Este cunoscut că proiecţia unui cerc pe un plan (planul cercului nefiind paralel cu planul de proiecţie) este o elipsă. În cazul indicat, elipsa ca proiecţie oblică a cercului se va construi cu ajutorul diametrilor conjugaţi, sau prin puncte, considerînd elipsa curbă afină a cercului.

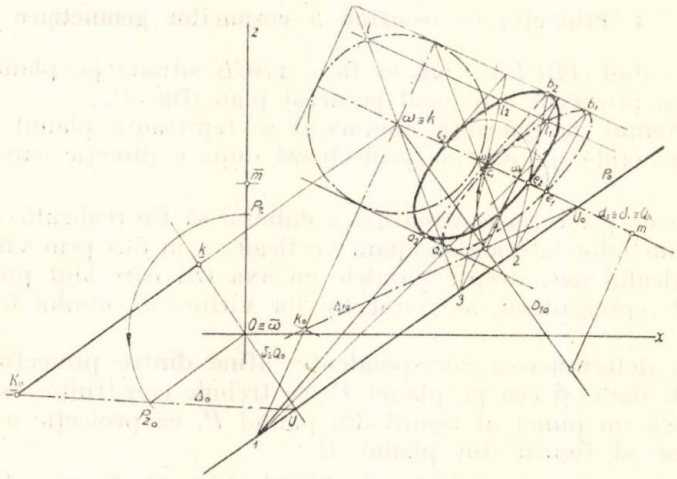

Fig. 3

Astfel, în fig. 3 s-a considerat cercul $\Omega(\omega, \bar{\omega})$ situat în planul reprezentării Π şi direcţia de proiectare $D_1(d_1, D_{10})$ faţă de planul $P(P_0, P_z)$. Aceasta este determinată de centrul cercului dat $\Omega(\omega, \bar{\omega})$ şi punctul $M(m, \bar{m})$. Se consideră prin proiectanta $D_1(d_1, D_{10})$ a centrului Ω un plan Q_1 proiectant faţă de planul Π al reprezentării, care se intersectează cu planul P după dreapta $\Delta_1(\delta_1, \Delta_{10})$. În rabatere, intersecţia dreptelor D_{10} şi Δ_{10} de-

termină punctul Ω_{10}, care este proiecţia oblică pe planul P a centrului Ω corespunzător cercului dat. Se proiectează ortogonal centrul Ω_{10} pe planul Π ducînd din Ω_{10} o perpendiculară la d_1; se găseşte astfel centrul proiectat ω_1, ca intersecţie dintre proiectanta (Ω_{10}, ω_1) şi proiecţia d_1.

Se consideră ab şi ce proiecţiile (pe planul Π al reprezentării) corespunzătoare diametrilor principali din cercul Ω, care se prelungesc pînă la axa de afinitate P_0, obţinîndu-se prin intersecţie cu aceasta punctele 1 şi U_0. Se observă că dreapta ce trece prin punctele 1 şi ω_1, intersectează proiectantele duse prin a şi b în a_1 şi b_1. În acest mod s-a obţinut proiecţia a_1b_1 pe planul Π, a diametrului AB.

Se trasează apoi coarda be, a cărei prelungire intersectează P_0 în punctul 2. Acest punct unit printr-o dreaptă cu proiecţia b_1, intersectează proiectanta punctului $e \equiv E$ în e_1, afinul celui dintîi. Se ia apoi $\omega_1 e_1 = \omega_1 c_1$, obţinînd astfel diametrii conjugaţi. Tot prin corespondenţă afină s-au găsit şi punctele $l_1 \equiv L_1$ ce rezultă din intersecţia cercului cu diagonalele pătratului circumscris acestuia.

Pentru a se obţine mărimea reală a proiecţiei oblice pe planul P a cercului considerat, se rabate, pe planul Π, planul P împreună cu elipsa proiecţie.

În acest scop, centrul Ω_{10} se rabate pe planul Π, axa de rabatere fiind urma P_6. Este suficient ca din punctul U_0, ca centru, să se traseze un arc de cerc, care intersectează proiectanta centrului $\omega \equiv \Omega$ în $\omega_2 \equiv \Omega_2$ şi care este centrul elipsei reale, ca proiecţie oblică a cercului dat.

Apoi, unind prin drepte punctele l cu ω_2 şi U_1 cu ω_2, se obţin direcţiile diametrilor conjugaţi rabătuţi în planul Π. Dreapta $l\omega_2$ se intersectează cu proiectantele punctelor $a \equiv A$ şi $b \equiv B$ în a şi în b, fiind unul din diametrii conjugaţi. Diametrul e_2c_2 se construieşte cu afinul lui e_1c_1. Astfel se poate trasa complet elipsa reprezentînd proiecţia reală a cercului dat.

4. Proiecţia centroafină a corpurilor geometrice

Se dă cubul $ABCEMNKL$ cu faţa $ABCE$ situată pe planul Π al reprezentării şi proiectat ortogonal pe acest plan (fig. 4).

În sistemul de referinţă plan-axial se reprezintă planul $P\,(P_0, P_z)$ de poziţie generală, pe care se proiectează după o direcţie oarecare cubul considerat.

Pentru ca proiecţia centroafină a cubului să fie realizată astfel încît proiecţiile muchiilor laterale să apară verticale, s-au dus prin vîrfurile proiecţiei poliedrului dat, drepte paralele cu axa Oz, care sînt proiecţiile pe planul Π al reprezentării, ale proiectatelor vîrfurilor cubului faţă de planul P.

Pentru determinarea corespondenţei afine dintre proiecţiile cubului pe planul Π, dată, şi cea pe planul P, ce trebuie construită, este necesar să se găsească un punct al figurii din planul P, ca proiecţie a punctului corespunzător al figurii din planul Π.

În acest scop, se consideră prin vîrful $A\,(a, \bar{a})$, dreapta $D_1\,(d_1, D_{10})$, care aşa cum se observă din figură, este determinată şi de punctul $R\,(r, R_0)$. Ea este direcţia de proiecţie (proiectare), adică $D_1\,(A, R)$.

Prin această dreaptă se consideră planul proiectant Q_1, care se intersectează cu planul P după dreapta $\Delta_1\,(\delta_1, \Delta_{10})$. Punctul de intersecţie dintre proiectanta D_1 şi dreapta Δ_1 este punctul căutat $A_1\,(a, A_{10})$. El se aduce în planul Π, rabătîndu-l împreună cu dreapta Δ_1 în jurul urmei

U_0, astfel încît arcul de cerc al rabaterii revine situat în planul Q_1. În r - batere s-a trasat arcul de cerc $A_{10}\,a_1$, cu centrul în U_0. Punctul A_1 $(a_1, \bar{a}_1)$ este afinul punctului A $(a, \bar{a})$.

Pentru obţinerea proiecţiei pe planul P a pătratului $ABCE$ ca faţă a cubului situată în planul Π, se construiesc elementele afine din planul P ca şi în cazul precedent.

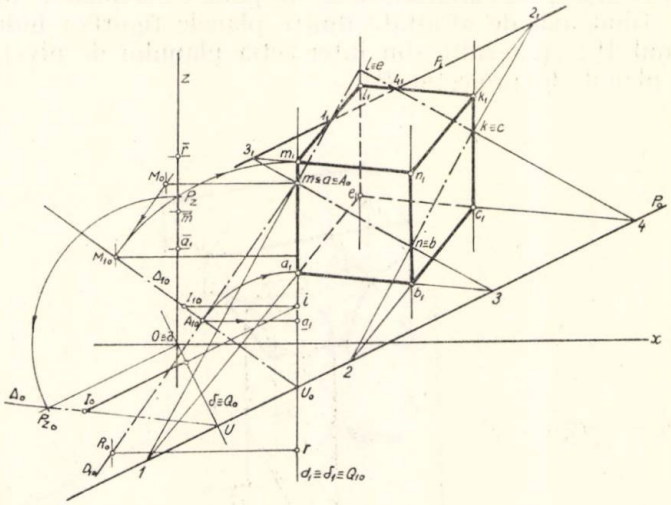

Fig. 4

Pentru construcţia proiecţiei centroafine a altei feţe, de exemplu $MNKL$, care se află în planul de nivel H de cotă egală cu muchia cubului dat, se rabate în planul Π muchia AM, axa de rabatere fiind urma Q_{10} a planului Q_1, astfel încît:

$$AM = \overline{Oa} = A_0M_0 = Z_M,$$

deoarece $Z_A = 0$.

Cum proiectantele rămîn paralele şi în rabatere cu direcţia de proiecţie, se duce din punctul M_0 rabătut o dreaptă paralelă cu dreapta D_{10}, care se intersectează cu (Δ_{10}) în punctul M_{10}. Acesta este ridicat şi rabătut pe planul Π, obţinînd proiecţia m_1 la intersecţia arcului de cerc de rabatere cu centrul în U_1 şi de rază U_1M_{10}.

Astfel, segmentul de dreaptă a_1m_1 este proiecţia centroafină pe planul P a muchiei AM, ca corespondentă afină a proiecţiei am din planul reprezentării aceleiaşi muchii.

Întrucît muchiile laterale sînt paralele între ele şi egale, proiecţiile lor pe planul P vor fi paralele şi de asemenea egale. De aceea: $a_1m_1 = b_1n_1 = c_1k_1 = e_1l_1$, ceea ce conduce la obţinerea proiecţiei centroafine

a feţei $MNKL$, care este paralelogramul $m_1n_1k_1l_1$, avînd laturile corespunzătoare egale şi paralele cu cele ale primului paralelogram proiecţie $a_1b_1c_1e_1$.

Considerînd vizibilitatea vîrfurilor în raport cu cotele comparate cu planul reprezentării Π, se construieşte conturul aparent al cubului (hexaedrului) dat, care este proiecţia centroafină a acestuia.

De observat că laturile omoloage ale pătratului $mnkl$ şi ale paralelogramului $m_1n_1k_1l_1$ se intersectează în puncte coliniare, dreapta respectivă P_{10} fiind axa de afinitate dintre planele figurilor indicate, rabătută în planul Π ; ea rezultă din intersecţia planului de nivel H al feţei $MNKL$ cu planul de proiecţie P.

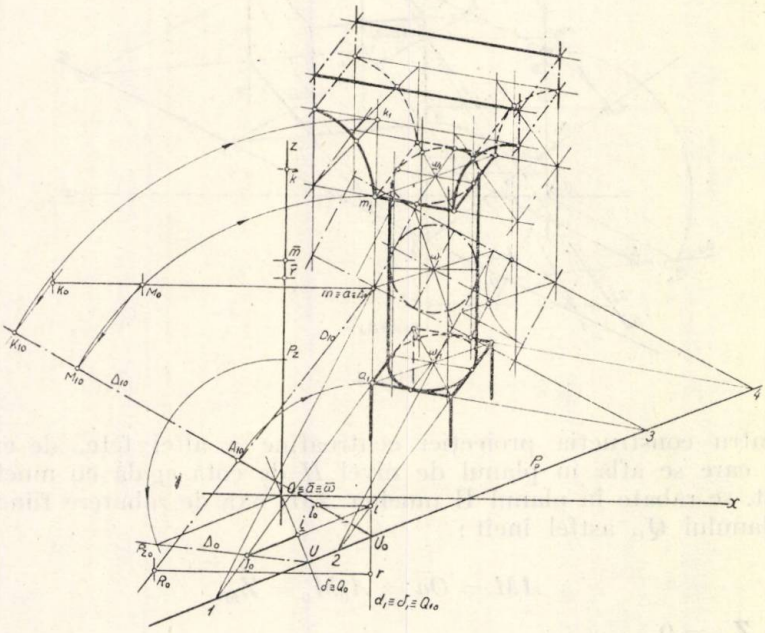

Fig. 5

Ca aplicaţie a reprezentării corpurilor geometrice, s-a considerat un element de construcţii format dintr-o coloană cilindrică verticală (suprafaţa cilindrică de revoluţie este perpendiculară pe planul Π al reprezentării). Cercul director $\Omega(\omega, \overline{\omega})$ este situat în planul Π, iar cercul superior limită este situat în planul de nivel de cotă $Z_H = AM = A_0M_0 = O\overline{m}$ (fig. 5).

În partea inferioară a cilindrului de revoluţie se află soclul coloanei de forma unei prisme pătrate drepte, ale cărei feţe laterale sînt tangente la suprafaţa cilindrică a coloanei, deci cercul Ω este înscris în pătratul de bază al prismei.

La partea superioară coloana susţine o grindă, care se termină pe reazem cu un pătrat circumscris cercului superior, cu laturile paralele cu ale pătratului din planul Π; pe distanţa dintre două coloane, grinda este în arc (semicerc), fiind plană la partea exterioară.

S-a construit proiecţia centroafină a prismei şi apoi a cilindrului înscris, utilizînd cele stabilite în cazul construcţiei proiecţiei centroafine a cubului.

Pentru construcţia elipselor, ca transformate afine ale cercurilor directoare Ω, s-au utilizat diametrii conjugaţi, precum şi diagonalele paralelogramelor circumscrise elipselor. Astfel, elipsele s-au trasat cu ajutorul punctelor de extremitate ale diametrilor conjugaţi şi de intersecţie dintre diagonale şi cercurile înscrise.

În partea inferioară s-au trasat laturile paralelogramului circumscris şi porţiuni din muchiile laterale, ţinînd seama de vizibilitatea în epură.

Pentru partea superioară a coloanei s-au construit de asemenea laturile poligonului circumscris, iar apoi proiecţiile centroafine ale sferturilor de cerc, care joacă rolul de curbe directoare ale cilindrilor de revoluţie, ale căror pînze limitează bolţile (arcele) dintre două coloane succesive.

Înălţimea la cheie, $\overline{mk} = Z_K - Z_M$, este raza semicercurilor bolţilor.

S-au găsit în epură proiecţiile centrelor şi ale punctelor în care elipsa, din care se detaşează bolta, este tangentă la laturile paralelogramului (ca proiecţie a pătratului circumscris).

Ţinînd seama de vizibilitatea în epură, s-a trasat conturul aparent al elementului de construcţie considerat, ca proiecţie centroafină a acestuia.

5. Concluzii

1. Construcţia proiecţiei paralele pe un plan P de poziţie generală a unui obiect, utilizînd ca direcţie de proiecţie o dreaptă de poziţie generală, s-a realizat pe baza corespondenţei afine dintre proiecţia obiectului de pe planul reprezentării şi proiecţia aceluiaşi obiect pe planul P de proiecţie.

2. Proiecţia centroafină oblică este utilizabilă în reprezentarea plană a obiectelor care interesează diverse domenii: construcţii, arhitectură, mecanică etc.; ea conduce la construcţia imaginii spaţiale a obiectului, permiţînd înţelegerea formei sale, a poziţiei într-un ansamblu şi mai ales perceperea laturii sale estetice.

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ZENTROAFFINE PROJEKTION MIT ALLGEMEINER PROJEKTIONS-RICHTUNG

(Zusammenfassung)

In vorliegendem Beitrag wurde ein Konstruktions-Verfahren der parallelen Projektion eines Körpers in einer Ebene $P(P_0\,;\,P_z)$ im axial-ebenen Referenz-System ausgearbeitet.

Grundsatz dieses Verfahrens bildet die Tatsache dass zwischen der orthogonalen Projektion in Ebene Π der Darstellung und der parallelen schiefen Projektion in der Projecktions-Ebene P eines gegebenen Körpers, gibt es eine affine Korrespondenz, was den Leitfaden des Verfahrens bildet.

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C. D. 532,5

ON A CLASS OF MOTIONS OF SIMPLE FLUIDS

BY

MARIAN ARON and VALERIU SAVA

1. Introduction

The aim of this work is to present some results in the dynamics of compressible simple fluids. The theory of these general fluids is based upon the following constitutive equation [1], [2],

$$\mathsf{T}(t) = -p(\rho)\mathsf{1} + \overset{\infty}{\underset{s=0}{\mathsf{H}}}(\mathsf{F}(s)\,;\ \rho). \tag{1.1}$$

Here T is the stress tensor, $p(\rho)$ a scalar function of the mass density ρ, $\mathsf{1}$—the unity tensor, $\mathsf{F}(s)$—the history of the relative deformation gradient and H—a functional. The tensor $\mathsf{F}(s)$ is the gradient of the relative deformation function

$$\hat{\mathsf{x}} = \mathsf{X}_t(\mathsf{x},\ \tau) \tag{1.2}$$

where $\hat{\mathsf{x}}$ is the position of the material point X at the time t and $\hat{\mathsf{x}}$ is the position of the same material point at the time τ, $\tau \leqslant t$[1]). The function X_t is the solution of the differential equation

$$\frac{d\hat{\mathsf{x}}}{d\tau} = \mathsf{v}(\hat{\mathsf{x}},\ \tau), \tag{1.3}$$

with the initial condition

$$\hat{\mathsf{x}}\,|_{\tau=t} = \mathsf{x}. \tag{1.4}$$

In the above $\mathsf{v} = \mathsf{v}(\mathsf{x},\ t)$ is an assigned velocity field.

The functional H can not be arbitrary. The principle of material objectivity requires that

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1) As usualy, we made the substitution $s = t - \tau$ and denote $F_t(\tau) \equiv F(s)$. The function $F(s)$ is called the history of the function $F_t(\tau)$.

$$\mathsf{Q}(0)\,\overset{\infty}{\underset{s=0}{\mathsf{H}}}\,(\mathsf{F}(s)\,;\ \rho)\,\mathsf{Q}(0)^T = \overset{\infty}{\underset{s=0}{\mathsf{H}}}\,(\mathsf{Q}(s)\mathsf{F}(s)\mathsf{Q}(0)^T\,;\ \rho) \tag{1.5}$$

for every continuous function $\mathsf{Q}(s)$ whose values are orthogonal tensors and every history $\mathsf{F}(s)$.

The relation (1.5) enable us to introduce a class of motions in the following section. Two non-isochoric flows which belong to this class will be studied in §§ **3** and **4**.

2. A Class of Flows

We say that a motion belongs to the class (P) if the corresponding stress field has two worthless shear stresses.

To make more clear this definition we shall consider some examples.

1. Let us consider the motion defined by the following velocity field, in an orthogonal Cartesian coordinate system $Oxyz$,

$$\dot{x} = \alpha x, \quad \dot{y} = \beta y, \quad \dot{z} = f(x). \tag{2.1}$$

Here α and β are arbitrary constants and f is an arbitrary function.

The components of the relative deformation gradient are given by

$$[\mathsf{F}(s)] = \begin{bmatrix} \mathrm{e}^{-\alpha s} & 0 & 0 \\ 0 & \mathrm{e}^{-\beta s} & 0 \\ \mathrm{e}^{-\alpha s}\displaystyle\int \frac{\mathrm{d}f(x\mathrm{e}^{-\alpha s})}{\mathrm{d}(x\mathrm{e}^{-\alpha s})}\,\mathrm{d}\tau - \frac{\mathrm{d}f(x)}{\mathrm{d}x} & 0 & 1 \end{bmatrix}. \tag{2.2}$$

Let us choose the orthogonal tensor Q^* with the components

$$[\mathsf{Q}^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

With this choise we have

$$[\mathsf{Q}^*]\,[\mathsf{F}(s)]\,[\mathsf{Q}^{*T}] = [\mathsf{F}(s)]. \tag{2.4}$$

Then, from (1.5) and (1.1), one easily results that the stress tensor T must satisfy

$$[\mathsf{Q}^*]\,[\mathsf{T}]\,[\mathsf{Q}^{*T}] = [\mathsf{T}], \tag{2.5}$$

such that we must have

$$t_{12} = t_{23} = 0. \tag{2.6}$$

2. Plane homogeneous motions whose velocity field is given by

$$\dot{x} = \alpha x + \beta y, \quad \dot{y} = \gamma x + \delta y, \quad \dot{z} = 0, \tag{2.7}$$

where α, β, γ, δ are arbitrary constants, lead us to the following matrix of components of the tensor $\mathsf{F}(s)$:

$$[\mathsf{F}(s)] = \left[\exp \left\{ -s \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \right]. \tag{2.8}$$

Hence the relations (2.7) defines a motion with constant strech history (see [3], sec. 109).

We choose now the orthogonal tensor whose components are given by

$$[\mathsf{Q}^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.9}$$

With this choise relations of the types (2.4) and (2.5) hold also such that we can infer

$$t_{13} = t_{23} = 0. \tag{2.10}$$

3. All viscometric flows belong also to the class (P) [1].

It is obvious that, for this class of flows, the equations of motions may be integrated such that the stress field is expressed with the aid of two arbitrary functions only. For the sake of simplicity, we shall illustrate this by the first two above examples, in the absence of the body forces. In the case of the first example one reduces the dynamical equations to

$$\begin{cases} t_{11,1} + t_{13,3} = \alpha^2 \rho x, \\ t_{22,2} = \beta^2 \rho y, \\ t_{13,1} + t_{33,3} = \alpha \rho x f'(x). \end{cases} \tag{2.11}$$

From (2.11) we obtain

$$\begin{cases} t_{11} = \Phi_{,33} + \alpha^2 \int\limits_{x_0}^{x} \rho x \, dx, \\ t_{22} = \varphi(x, z, t) + \beta^2 \int\limits_{y_0}^{y} \rho y \, dy, \\ t_{33} = \Phi_{,11} + \alpha x f'(x) \int\limits_{z_0}^{z} \rho \, dz, \\ t_{13} = -\Phi_{,13}, \end{cases} \tag{2.12}$$

where $\Phi(x, y, z, t)$ and $\varphi(x, z, t)$ are arbitrary functions.

In the case of plane homogeneous motions the dynamical equations are of the form

$$(2.13)\qquad \begin{cases} t_{11,1} + t_{12,2} = \rho[(\alpha^2 + \beta\gamma)x + \beta(\alpha + \delta)y], \\ t_{12,1} + t_{22,2} = \rho\,[\gamma(\alpha + \delta)x + (\alpha\beta + \delta^2)y], \\ t_{33,3} = 0, \end{cases}$$

such that we can infer

$$(2.14)\qquad \begin{cases} t_{11} = \Phi_{,22} + (\alpha^2 + \beta\gamma)\displaystyle\int_{x_0}^{x} \rho x\,\mathrm{d}x + \beta(\gamma + \delta)y\int_{x_0}^{x} \rho\,\mathrm{d}x, \\ t_{22} = \Phi_{,11} + \gamma(\alpha + \delta)x\displaystyle\int_{y_0}^{y} \rho\,\mathrm{d}y + (\alpha\beta + \delta^2)\int_{y_0}^{y} \rho y\,\mathrm{d}y, \\ t_{12} = -\Phi_{,12}, \\ t_{33} = \varphi(x, y, t), \end{cases}$$

where $\Phi(x, y, z, t)$ and $\varphi(x, y, t)$ are arbitrary functions.

3. The First Example

In this section we shall deal with the motion whose components of the velocity field are given, in an orthogonal Cartesian coordinate system $Oxyz$, by

$$(3.1)\qquad \dot{x} = \alpha x, \qquad \dot{y} = 0, \qquad \dot{z} = f(x).$$

From (2.12), (2.2) and (1.1) we have

$$(3.2)\qquad \begin{cases} \Phi_{,33} + \alpha^2\displaystyle\int_{x_0}^{x} \rho x\,\mathrm{d}x = -p(\rho) + \overset{\infty}{\underset{s=0}{h}}_{11}(\alpha, f(.), \rho), \\ \Phi_{,11} + \alpha x f'(x)\displaystyle\int_{z_0}^{z} \rho\mathrm{d}z = -p(\rho) + \overset{\infty}{\underset{s=0}{h}}_{33}(\alpha, f(.), \rho), \\ -\Phi_{,13} = \overset{\infty}{\underset{s=0}{h}}_{13}(\alpha, f(.), \rho), \qquad \varphi(x, z, t) = -p(\rho) + \overset{\infty}{\underset{s=0}{h}}_{22}(\alpha, f(.), \rho), \end{cases}$$

where $\overset{\infty}{\underset{s=0}{h}}_{ij}(\alpha, f(.), \rho)$, $(i, j = 1, 2, 3)$ are the components of the response functional $\mathsf{H}(\alpha, f(.), \rho)$.

From the last of equations (3.2), we obtain

$$\left(-\frac{\partial p}{\partial \rho} + \frac{\partial \overset{\infty}{\underset{s=0}{h}}_{22}}{\partial \rho}\right)\frac{\partial \rho}{\partial y} = 0. \tag{3.3}$$

We shall suppose that

$$\frac{\partial \rho}{\partial y} = 0. \tag{3.4}$$

In this case the system (3.2) has a solution if and only if

$$\begin{cases} \overset{\infty}{\underset{s=0}{h}}_{11}(\alpha, f(.), \rho) = p(\rho) + \alpha^2 \int\limits_{x_0}^{x} \rho x \, dx - \int\limits_{x_0}^{x} \overset{\infty}{\underset{s=0}{h}}_{13,3} \, dx, \\ \overset{\infty}{\underset{s=0}{h}}_{33}(\alpha, f(.), \rho) = p(\rho) + \alpha x f'(x) \int\limits_{z_0}^{z} \rho \, dz - \int\limits_{z_0}^{z} \overset{\infty}{\underset{s=0}{h}}_{13,1} \, dz. \end{cases} \tag{3.5}$$

We have now, by (1.1),

$$\begin{cases} t_{11} = \alpha^2 \int\limits_{x_0}^{x} \rho x \, dx - \int\limits_{x_0}^{x} \overset{\infty}{\underset{s=0}{h}}_{13,3} \, dx, \\ t_{22} = -p(\rho) + \overset{\infty}{\underset{s=0}{h}}_{22}(\alpha, f(.), \rho), \\ t_{33} = \alpha x f'(x) \int\limits_{z_0}^{z} \rho \, dz + \int\limits_{z_0}^{z} \overset{\infty}{\underset{=0}{h}}_{13,1} dz, \quad t_{13} = \overset{\infty}{\underset{s=0}{h}}_{13}(\alpha, f(.), \rho). \end{cases} \tag{3.6}$$

The function $\rho = \rho(x, z, t)$ is the solution of the continuity equation

$$\frac{\partial \rho}{\partial t} + \alpha x \frac{\partial \rho}{\partial x} + f(x) \frac{\partial \rho}{\partial z} + \alpha \rho = 0. \tag{3.7}$$

Let us suppose that

$$f(x) \equiv \alpha x \tag{3.8}$$

and the fluid flows between two parallel plates

$$x - z = \pm C. \tag{3.9}$$

The volume discharge per unit time through a cross - section of the channel of unit depth is

$$Q = 2\alpha \int_{\frac{D-C}{2}}^{\frac{D+C}{2}} (C - z)\, \mathrm{d}z = \frac{\alpha C}{2}(4C - 3D), \tag{3.10}$$

if $x + z = D$ is the equation of the cross-section. Thus we get

$$\alpha = \frac{2Q}{C(4C-3D)}. \tag{3.11}$$

In this case the solution of the continuity equation is given by

$$\rho = \rho(x\mathrm{e}^{-\alpha t}, z - x(1 - \mathrm{e}^{\alpha t}))\mathrm{e}^{-\alpha t}. \tag{3.12}$$

4. Plane Homogeneous Motions

From (2.14), (2.8) and (1.1) we have

$$\left\{\begin{aligned}
\Phi_{,22} &= -p(\rho) + h_{11}(\alpha, \beta, \gamma, \delta, \rho) - (\alpha^2 + \beta\gamma)\int_{x_0}^{x} \rho x\, \mathrm{d}x - \\
&\quad - \beta(\gamma + \delta) y \int_{x_0}^{x} \rho\, \mathrm{d}x, \\
\Phi_{,11} &= -p(\rho) + h_{22}(\alpha, \beta, \gamma, \delta, \rho) - \gamma(\alpha + \delta) x \int_{y_0}^{y} \rho\, \mathrm{d}y - \\
&\quad - (\alpha\beta + \delta^2)\int_{y_0}^{y} \rho y\, \mathrm{d}y, \\
\Phi_{,12} &= -h_{12}(\alpha, \beta, \gamma, \delta, \rho), \\
\varphi(x, y, t) &= -p(\rho) + h_{33}(\alpha, \beta, \gamma, \delta, \rho).
\end{aligned}\right. \tag{4.1}$$

In the same way as in the previous section we can obtain

$$\frac{\partial \rho}{\partial z} = 0 \tag{4.2}$$

and the conditions

$$(4.3)\quad \begin{cases} h_{11} = p(\rho) + (\alpha^2 + \beta\gamma)\displaystyle\int_{x_0}^{x} \rho x \, dx + \beta(\gamma + \delta) y \int_{x_0}^{x} \rho \, dx - \int_{x_0}^{x} h_{12,2} \, dx, \\ h_{22} = p(\rho) + \gamma(\alpha + \delta) x \displaystyle\int_{y_0}^{y} \rho \, dy + (\alpha\beta + \delta^2) \int_{y_0}^{y} \rho y \, dy - \int_{y_0}^{y} h_{12,1} \, dy. \end{cases}$$

The stress field one reads off from (2.14), (4.3) and (1.1).

We suppose that

$$(4.4)\qquad \alpha = \beta = \gamma = \delta,$$

and consider $x - y = \pm C$ to be two fixed planes between the fluid flows. If $x + y = D$ is the equation of the cross-section we obtain

$$(4.5)\qquad \alpha = -\frac{Q}{DC\sqrt{2}}.$$

The continuity equation yields

$$(4.6)\qquad \rho = \rho(x + y)e^{-2\alpha t}.$$

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ASUPRA UNEI CLASE DE MIŞCĂRI ALE FLUIDELOR SIMPLE

(Rezumat)

Se introduce o clasă de mişcări şi se studiază două curgeri neizocore, ale fluidelor simple compresibile, care aparţin acestei clase.

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MECANICĂ TEORETICĂ. FIZICĂ

TRANSFORMATIONS POLYTROPIQUES HYPERSONIQUES D'UN FLUIDE IDÉAL (II)

PAR

C. L. ACKER

En continuant les observations sur les transformations polytropiques hypersoniques (v. [3]), remarquons tout d'abord que l'équation d'Euler prend la forme

$$\bar{a} - \bar{\Phi} = wb \operatorname{grad} V_s, \tag{1}$$

ou

$$\bar{a} - \bar{\Phi} = - c^2 \operatorname{grad} \ln \rho \tag{2}$$

($\bar{a}$ est l'accélération, $\bar{\Phi}$ la force massique, w — le rapport des chaleurs spécifiques C_p/C_v , b — la pression, V_s — le volume spécifique, c — la vitesse locale du son, ρ — la densité).

1. Lorsque le repère $T\,(M;\, I_1,\, I_2,\, I_3)$ que nous avons utilisé dans [3] (ayant l'origine M à un point quelconque du domaine D occupé par le fluide et $I_3 = \bar{v}/v$ ($\bar{v}$ est le vecteur vitesse)) sera canonisé tel que I_2 soit normal au plan ($\bar{v}$, $\bar{\Phi}$), c'est-à-dire ($\bar{\Phi}.I_2 =$) $\Phi_2 = 0$, alors, en projetant l'équation (1) sur l'axe I_2, on constate qu'à un moment quelconque donné [1])

$$\frac{\mathrm{d} \ln V_s/\mathrm{d}s_2}{1/R^{(2)}_{n\mathrm{tr}}} \geqslant K^2, \tag{3}$$

[1]) Les équations (s) et (h) de [3 p. 135], qui caractérisent le régime supersonique et le régime hypersonique, doivent être considérées de la forme $v = (1 ,\ldots + F^2) \sqrt{C^* w \rho^{w-1}}$ respectivement $v = (K + F^2) \sqrt{C^* w \rho^{w-1}}$, l'équation de l'état thérmodynamique étant $b = C^* \rho^w$ (b est la pression).

($\mathrm{d}/\mathrm{d}s_2$ indique la dérivé le long de la normale au plan $(\bar{v}, \bar{\Phi})$, $1/R^{(2)}_{n\,\mathrm{tr}}$ est la courbure normale de la configuration Myller (tr., I_3, P_2)[2]) et K est une des plus faibles valeurs du nombre local de Mach correspondant au régime hypersonique),

$$\frac{I_2 \operatorname{grad} \ln V_s}{1/R^{(2)}_{n\mathrm{tr}}} \geqslant K^2 . \tag{3'}$$

S'il s'agit tout simplement d'une transformation polytropique supersonique quelconque, alors

$$\frac{\mathrm{d} \ln V_s/\mathrm{d}s_2}{1/R^{(2)}_{n\mathrm{tr}}} > 1 , \tag{4}$$

ou

$$\frac{I_2 \operatorname{grad} \ln V_s}{1/R^{(2)}_{n\mathrm{tr}}} > 1 . \tag{4'}$$

2. En projetant l'équation (2) sur le même axe I_2, on a

$$-R^{(2)}_{n\mathrm{tr}} \frac{\mathrm{d} \ln \rho}{\mathrm{d}s_2} \geqslant K^2 \tag{5}$$

pour les transformations polytropiques hypersoniques, respectivement

$$-\frac{I_2 \operatorname{grad} \ln \rho}{1/R^{(2)}_{n\mathrm{tr}}} > 1 \tag{6}$$

pour les transformations polytropiques supersoniques quelconques.

3. Canonisons maintenant le repère T tel que I_2 soit normal au plan $(\bar{v}, \operatorname{grad} \rho)$, c'est à dire $\rho_2 = 0$. En projetant alors l'équation (1) sur la direction $(\bar{v} \times \operatorname{grad} \rho)$, on constate que pendant la transformation polytropique hypersonique d'un fluide idéal

$$\frac{\Phi_2}{1/R^{(2)}_{n\mathrm{tr}}} \geqslant c^2 K^2 , \tag{7}$$

tandis que pour une transformation polytropique supersonique quelconque on a

$$\frac{\Phi_2}{1/R^{(2)}_{n\mathrm{tr}}} > c^2 \tag{8}$$

($1/R^{(2)}_{n\mathrm{tr}}$ est la courbure normale de la configuration Myller (tr., I_3, P_2), P_2 étant cette fois le plan $(\bar{v}, \operatorname{grad} \rho)$).

[2]) tr. signifie la trajectoire de la particule et P_2 est le plan $(\bar{v}, \bar{\Phi})$.

Observation. Chacune des propriétés précédentes (obs. 1, 2, 3) caractérise les transformations hypersoniques (respectivement les transformations supersoniques) dans la classe des transformations polytropiques générales d'un fluide idéal.

4. Lorsqu'il s'agit de la classe particulière des transformations PH[3]) pour laquelle la grandeur de la vitesse est stationnaire ($v_t = 0$; v. [3, Cl. 2]), alors

$$\frac{\Phi_3 - V_s b_3}{(\ln v)_3} \geqslant c^2 K^2, \tag{9}$$

$$\Phi_3 = \Phi I_3, \quad b_3 = I_3 \operatorname{grad} b, \quad (\ln v)_3 = I_3 \operatorname{grad} \ln v.$$

On observe que la grandeur de la vitesse croît à un déplacement δ dans la direction de $\bar{v}$, si on a

$$\Phi_3 > V_s b_3,$$

ce qui entraîne aussi

$$\Phi_3 \geqslant V_s b_3 + c^2 K^2 (\ln v)_3.$$

Par contre, à un même déplacement δ, la grandeur de la vitesse abaisse si on a

$$\Phi_3 < V_s b_3,$$

ce qui entraîne aussi

$$\Phi_3 \leqslant V_s b_3 + c^2 K^2 (\ln v)_3.$$

5. Pour la classe 3 (v. [3]) des transformations PH pour lesquelles la dérivé substantielle de la grandeur de la vitesse est nulle ($v_t + v v_3 = 0$) signalons les relations

$$\frac{\mathrm{d}V_s/\mathrm{d}s_3}{\Phi_3} \geqslant \frac{K^2}{v^2 \rho}, \tag{10}$$

$$\frac{\rho_3}{\Phi_3} \geqslant \frac{K^2}{v^2 V_s}, \tag{11}$$

où

$$\frac{\mathrm{d}V_s}{\mathrm{d}s_3} = I_3 . \operatorname{grad} V_s, \quad \rho_3 = I_3 \operatorname{grad} \rho.$$

Évidemment, si $\Phi_3 > 0$ alors la densité croît à un déplacement δ (respectivement le volume spécifique abaisse) et par contre, si $\Phi_3 < 0$ alors à un déplacement δ, la valeur de la densité décroît.

[3]) Polytropiques hypersoniques.

6. Pendant les transformations PH de la Cl.4 ($v_3 = 0$)[4]) qui se développent tel que les lignes du courant restent isotaheïques, signalons

$$\frac{\mathrm{d}V_s/\mathrm{d}s_3}{v_t - \Phi_3} \geqslant \frac{K^2}{v^2\rho}, \tag{12}$$

ou

$$\frac{\mathrm{d}\ln\rho/\mathrm{d}s_3}{\Phi_3 - v_t} \geqslant \frac{K^2}{v^2}, \tag{13}$$

qui indiquent un accroissement du volume spécifique (respectivement un abaissement de la densité) à un déplacement δ, lorsque $v_t > \Phi_3$, et par contre, l'abaissement du volume spécifique, au même déplacement δ, si on a $v_t < \Phi_3$ (v_t est la dérivé partielle par rapport au temps, de la grandeur de la vitesse).

7. Pendant le régime hypersonique des transformations polytropiques pour lesquelles la grandeur de la vitesse a la même valeur à tous les points du domaine D, à un moment quelconque donné (grad $v = 0$), on a de plus (sauf la propriété précédente, obs. 6)

$$\frac{(\mathrm{rot}\,\bar{v}_{I_1 I_2})^2}{k_{l.c}^2} \geqslant c^2 K^2, \tag{14}$$

où $(\mathrm{rot}\,\bar{v})_{I_1 I_2}$ est la composante de $(\mathrm{rot}\,\bar{v})$ orthogonale à la vitesse et $k_{l.c}$ est la courbure de la ligne du courant.

8. Si $v = \text{const.}$, remarquons aussi

$$\frac{\mathrm{d}\ln\rho/\mathrm{d}s_3}{\Phi_3} \geqslant \frac{K^2 k_m^2(\bar{v}/v)}{(\mathrm{d}\ln\rho/\mathrm{d}t)^2}, \tag{15}$$

ou

$$\frac{\mathrm{d}(\rho^2)/\mathrm{d}s_3}{\Phi_3} \geqslant \frac{2K^2 k_m^2(\bar{v}/v)}{(\mathrm{d}V_s/\mathrm{d}t)^2} \tag{16}$$

($\mathrm{d}/\mathrm{d}t$ indique la dérivé substantielle et $k_m(\bar{v}/v)$[5]) est la courbure moyenne du champ $(\bar{v}/v)$, à un moment quelconque donné et à un point quelconque donné de D).

9. Pour la classe 7 des transformations PH pendant lesquelles la dérivé substantielle de la densité est nulle ($\rho_t + v\,\rho_3 = 0$), il est à remarquer aussi

$$-\frac{\rho_t}{\Phi_3 - \mathrm{d}v/\mathrm{d}t} \geqslant \frac{K^2\rho}{v} \tag{17}$$

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[4]) Pour cette classe, dans [3, pp. 138—139, obs. 7], K doit être remplacé par K^2.

[5]) Dans [3, p. 138, obs. 4, 6 et 7], $R_m(\bar{v}/v)$ signifie la courbure moyenne du champ $(\bar{v}/v)$.

(ρ_t est la dérivé partielle de la densité par rapport au temps). Lorsque $\Phi_3 > \mathrm{d}v/\mathrm{d}t$ à un point quelconque de D, alors la valeur de la densité abaisse à ce point ; par contre si $\Phi_3 < \mathrm{d}v/\mathrm{d}t$, alors la densité croît à ce point.

10. Retenons la Cl. 8 ($\rho_3 = 0$) des transformations PH pour les propriétés

$$\frac{\Phi_3}{\mathrm{d}\ln v/\mathrm{d}s_{\mathrm{tr}}} \geqslant c^2 K^2 \tag{18}$$

et

$$\frac{(\ln \rho)_t + v_3}{k_m(\bar{v}/v)} \geqslant cK \tag{19}$$

($\mathrm{d}/\mathrm{d}s_{\mathrm{tr}}$ indique la dérivé le long de la trajectoire de la particule et $v_3 = I_3 \operatorname{grad} v$).

11. Sauf la propriété (18), signalons aussi, pour la Cl. 9 ($\operatorname{grad} \rho = 0$) des transformations PH, la propriété

$$\frac{\mathrm{d}I_3}{\mathrm{d}s_{\mathrm{tr}}} \cdot \bar{\Phi}_{I_1 I_2} = c^2 K^2 \left(\frac{\mathrm{d}I_3}{\mathrm{d}s_{\mathrm{tr}}}\right)^2 \tag{20}$$

($\mathrm{d}I_3/\mathrm{d}s_{\mathrm{tr}}$ est le vecteur de la courbure de la trajectoire d'une particule et $\bar{\Phi}_{I_1 I_2}$ est la composante normale à la vitesse, de la force massique).

12. Enfin s'il s'agit de la Cl. 10 ($\rho = \text{const.}$) des transformations PH, ajouttons

$$\frac{\mathrm{d}(v^2)/\mathrm{d}s_3}{k_m(v/v)} \geqslant 2c^2 K^2. \tag{21}$$

Tous les propriétés indiquées au-dessus (obs. 4...12) caractérisent le régime hypersonique dans les classes particulières des transformations polytropiques considérées.

Les relations (9)...(21), considérées comme des inégalités strictes, pour $K = 1$, nous offrent des propriétés caractéristiques du régime supersonique, pendant les transformations polytropiques particulières respectives.

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TRANSFORMĂRI POLITROPICE HIPERSONICE ALE UNUI FLUID IDEAL (II)

(Rezumat)

Sînt prezentate proprietăţi ale transformărilor politropice hipersonice generale ale unui fluid ideal, precum şi proprietăţi ale unor clase particulare de transformări politropice hipersonice (cu referiri la regimul supersonic oarecare), asemenea proprietăţi fiind stabilite prin folosirea teoriei sistemelor diferenţiale exterioare şi a metodei reperului mobil al lui E. Cartan.

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D.C. 539.16 ; 519.4

THE EXTENSION FORMULAS FOR $SL(4, C)$

BY

GH. ZET

1. Introduction

Recent developments indicate that unitary infinite dimensional representations of certain noncompact Lie groupe may be usefully employed in describing the physics of elementary particles. One of the ways in which noncompact groups are being used in elementary particle physics is the formation of supermultiplets containing an infinite number of „particles". One assumes that particles are grouped into finite-dimensional multiplets ; then an infinite number of these finite-dimensional multiplets are needed to „fill up" the infinite dimensional supermultiplet which is generated by the noncompact group. The main problems of the representations theory for a given noncompact group G are :

1) What are the irreducible unitary representations of G.

2) What is the reduction of an irreducible representation of G when restricted to the maximal compact subgroup or to any other sufficiently large subgroup.

It should be pointed out that for compact groups (at least compact Lie groups) there is a rather satisfactory answer to these questions ; for noncompact groups however, we are only now beginning to see the patterns that emerge.

A basic construction by means of which it is possible to construct irreducible unitary representations of large classes of groups is based on the notion of induced representations. For certain classes of groups all the irreducible unitary representations can be constructed as induced representations, while for other classes this is the case for a large number of irreducible unitary representations.

The problem of reduction to the maximal compact subgroup is completely solved for the groups semidirect product [1]. In this paper we describe a method for solving this problem for a principal degenerate series of unitary irreducible representations of $SL(4, C)$. Namely, the generators of $SL(4,C)$ are written as polynomial functions (the „extension formulas") of the generators of suitably chosen unitary representations of $SU(4) \times) T_{15}$ (semidirect product), where T_{15} stands for a set of Abelian generators transforming under $SU(4)$ as a tensor belonging to the 15-dimensional adjoint representation of $SU(4)$. Since the representations of $SU(4)\times)$

T_{15} in a $SU(4)$ basis are known [1], this provides a solution to the reduction problem in setting up the $SL(4, C)$ representations in an $SU(4)$ basis. The extension formulas are in this case just the Gell-Mann formulas [2]. In [3] it is shown that Gell-Mann formulas applies to the maximally degenerate series of unitary irreducible representations of $SL(n, C)$.

2. The Principal Degenerate Series of $SL(4, C)$

The group $SL(4, C)$ is the group of all unimodular complex matrices in four dimensions. The Lie algebra of this group is 30-dimensional, being spanned by the generators M^{α}_{β} ($\alpha, \beta = 1, 2, 3, 4$ and $M^{\alpha}_{\alpha} = 0$) of its maximal compact subgroup $SU(4)$ and by „noncompact" generators N^{α}_{β} ($\alpha, \beta = 1, 2, 3, 4$ and $N^{\alpha}_{\alpha} = 0$). (The sum convention over repeated indices is made). The commutation relations in a unitary representation of $SL(4, C)$ are:

$$[M^{\alpha}_{\beta}, M^{\lambda}_{\mu}] = \delta^{\alpha}_{\mu} M^{\lambda}_{\beta} - \delta^{\lambda}_{\beta} M^{\alpha}_{\mu}, \tag{2.1 a}$$

$$[M^{\alpha}_{\beta}, N^{\lambda}_{\mu}] = \delta^{\alpha}_{\mu} N^{\lambda}_{\beta} - \delta^{\lambda}_{\beta} N^{\alpha}_{\mu}, \tag{2.1 b}$$

$$[N^{\alpha}_{\beta}, N^{\lambda}_{\mu}] = \delta^{\lambda}_{\beta} M^{\alpha}_{\mu} - \delta^{\alpha}_{\mu} M^{\lambda}_{\beta}, \tag{2.1 c}$$

where δ^{α}_{β} is the Kronecker symbol. In addition, the followings Hermiticity properties are satisfied:

$$(M^{\alpha}_{\beta})^{+} = M^{\beta}_{\alpha}, \qquad (N^{\alpha}_{\beta})^{+} = N^{\beta}_{\alpha}. \tag{2.2}$$

To get the representations in the principal degenerate series, we pick a set of $r < 4$ integers $n_1, n_2, \ldots, n_r$ such that $n_1 + n_2 + \ldots + n_r = 4$. In this paper we consider the case $r = 2$, $n_1 = n_2 = 2$; the case $r = 2$, $n_1 = 3$, $n_2 = 1$ has been treated in a previous work [3]. We writte the matrix $g \in SL(4, C)$ in the form:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \tag{2.3}$$

where $g_{11}, g_{12}, g_{21}, g_{22}$ are 2×2 matrices. Define two subgroups K and Z of $SL(4, C)$:

$$K \ni k = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix}, \quad \det k = 1, \quad Z \ni z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & \beta & 1 & 0 \\ \lambda & \omega & 0 & 1 \end{pmatrix}, \tag{2.4}$$

where k_{11}, k_{12}, k_{22} are 2×2 matrices, too. Then, almost every element $g \in SL(4, C)$ can be expressed in a unique way as the product of one element in K and one element in Z [4]:

$$g = kz, \qquad k \in K, \qquad z \in Z. \tag{2.5}$$

In particular, the element $zg \in SL(4, C)$ may be written as:

$$zg = kz', \qquad z' \in Z. \tag{2.6}$$

The representations of the series ($r = 2$, $n_1 = n_2 = 2$) are constructed in the Hilbert space $L^2(Z)$ of all functions $f(z)$ which are square-integrable, i. e.:

$$\int |f(z)|^2 d\mu(z) < \infty ,$$

where $\mu(z)$ is an invariant measure on Z and $d\mu(z)$ is the product

$$d(\mathcal{R}e\ \alpha) \cdot d(\mathcal{I}m\ \alpha) \cdot d(\mathcal{R}e\ \beta) \cdot d(\mathcal{I}m\ \beta) \cdot d(\mathcal{R}e\ \gamma) \cdot d(\mathcal{I}m\ \gamma) \cdot d(\mathcal{R}e\ \omega) \cdot d(\mathcal{I}m\ \omega).$$

The representations $g \to R(g)$ of our series are given by formula:

$$R(g) f(z) = |\Lambda_2|^{m + i\rho - 4} (\Lambda_2)^{-m} f(z'), \tag{2.7}$$

where $\Lambda_2 = \det k_{22}$ and z' are determinated by (2.6). Here, m is an arbitrary integer number and ρ an arbitrary real number.

3. The Extension Formulas

The formula (2.7) has the disadvantage that its right-hand side has a dependence on the continuous parameter ρ, when g is in $SU(4)$. Or, it is known that unitary irreducible representations of $SU(4)$ are characterized by discrete indices only. This dependence can be removed if we define a new set of basis vectors $|z)$:

$$|z) = \exp[i\Phi(z)]\, |z>, \tag{3.1}$$

where $|z> \equiv |\alpha, \beta, \gamma, \omega> \equiv f(z)$ and

$$\Phi(z) = -\frac{\rho}{2} \ln(1 + |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\omega|^2 + |\alpha\omega - \beta\gamma|^2). \tag{3.2}$$

Then, (2.7) become:

$$R(g)\,|z) = \exp[i\Phi(z) - i\Phi(z')]\, |\Lambda_2|^{m + i\rho - 4} (\Lambda_2)\, |^{-m} |z'). \tag{3.3}$$

To obtain the infinitesimal generators of this representation, we consider g in (3.3) to be an element close to the identity. We list now the generators M^{α}_{β} and N^{α}_{β}:

$$
(3.4)\quad \begin{cases}
M_1^1 = \dfrac{m}{2} + \alpha\dfrac{\partial}{\partial\alpha} + \gamma\dfrac{\partial}{\partial\gamma} - \alpha^*\dfrac{\partial}{\partial\alpha^*} - \gamma^*\dfrac{\partial}{\partial\gamma^*}, \\[2ex]
M_1^2 = \alpha\dfrac{\partial}{\partial\beta} + \gamma\dfrac{\partial}{\partial\omega} - \beta^*\dfrac{\partial}{\sigma\alpha^*} - \omega^*\dfrac{\partial}{\partial\gamma^*}, \\[2ex]
M_1^3 = -\dfrac{m}{2}\alpha - 2\alpha - \alpha^2\dfrac{\partial}{\partial\alpha} - \alpha\beta\dfrac{\partial}{\partial\beta} - \alpha\gamma\dfrac{\partial}{\partial\gamma} - \gamma\beta\dfrac{\partial}{\partial\omega} - \dfrac{\partial}{\partial\alpha^*}, \\[2ex]
M_1^4 = -\dfrac{m}{2}\gamma - 2\gamma - \alpha\gamma\dfrac{\partial}{\partial\alpha} - \alpha\omega\dfrac{\partial}{\partial\beta} - \gamma^2\dfrac{\partial}{\partial\gamma} - \gamma\omega\dfrac{\partial}{\partial\omega} - \dfrac{\partial}{\partial\gamma^*}, \\[2ex]
M_2^2 = \dfrac{m}{2} + \beta\dfrac{\partial}{\partial\beta} + \omega\dfrac{\partial}{\partial\omega} - \beta^*\dfrac{\partial}{\partial\beta^*} - \omega^*\dfrac{\partial}{\partial\omega^*}, \\[2ex]
M_2^3 = -\dfrac{m}{2}\beta - 2\beta - \alpha\beta\dfrac{\partial}{\partial\alpha} - \beta^2\dfrac{\partial}{\partial\beta} - \alpha\omega\dfrac{\partial}{\partial\gamma} - \beta\omega\dfrac{\partial}{\partial\omega} - \dfrac{\partial}{\partial\beta^*}, \\[2ex]
M_2^4 = -\dfrac{m}{2}\omega - 2\omega - \beta\gamma\dfrac{\partial}{\partial\alpha} - \beta\omega\dfrac{\partial}{\partial\beta} - \gamma\omega\dfrac{\partial}{\partial\gamma} - \omega^2\dfrac{\partial}{\partial\omega} - \dfrac{\partial}{\partial\omega^*}, \\[2ex]
M_3^3 = -\dfrac{m}{2} + \alpha\dfrac{\partial}{\partial\alpha} - \beta\dfrac{\partial}{\partial\beta} + \alpha^*\dfrac{\partial}{\partial\alpha^*} + \beta^*\dfrac{\partial}{\partial\beta^*}, \\[2ex]
M_3^4 = -\gamma\dfrac{\partial}{\partial\alpha} - \omega\dfrac{\partial}{\partial\beta} + \alpha^*\dfrac{\partial}{\partial\gamma^*} + \beta^*\dfrac{\partial}{\partial\omega^*};
\end{cases}
$$

$$
(3.5)\quad N_\beta^\alpha = \rho P_\beta^\alpha + N_\beta^\alpha,
$$

where

$$
\begin{cases}
P_1^1 = \dfrac{1}{2A}(|\alpha|^2 + |\gamma|^2 + |\alpha\omega - \beta\gamma|^2 - |\beta|^2 - |\omega|^2 - 1), \\[2ex]
P_1^2 = \dfrac{1}{A}(\alpha\beta^* + \gamma\omega^*), \\[2ex]
P_1^3 = \dfrac{1}{A}(\alpha + \omega^*(\alpha\omega - \beta\gamma)), \\[2ex]
P_1^4 = \dfrac{1}{A}(\gamma - \beta^*(\alpha\omega - \beta\gamma)),
\end{cases}
$$

$$
\left\{
\begin{aligned}
&P_2^2 = \frac{1}{2A}\left(|\beta|^2 + |\omega|^2 + |\alpha\omega - \beta\gamma|^2 - |\alpha|^2 - |\gamma|^2 - 1\right),\\
&P_2^3 = \frac{1}{A}\left(\beta + \gamma^*(\alpha\omega - \beta\gamma)\right),\\
&P_2^4 = \frac{1}{A}\left(\omega + \alpha^*(\alpha\omega - \beta\gamma)\right),\\
&P_3^3 = \frac{1}{2A}\left(1 + |\gamma|^2 + |\omega|^2 - |\alpha|^2 - |\beta|^2 - |\alpha\omega - \beta\gamma|^2\right),\\
&P_3^4 = -\frac{1}{A}\left(\alpha^*\gamma + \beta^*\omega\right),\ A = 1 + |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\omega|^2 +\\
&\quad + |\alpha\omega - \beta\gamma|^2;\\
&\overline{N}_1^1 = -2\mathrm{i} - \mathrm{i}\left(\alpha\frac{\partial}{\partial\alpha} + \gamma\frac{\partial}{\partial\gamma} + \alpha^*\frac{\partial}{\partial\alpha^*} + \gamma^*\frac{\partial}{\partial\gamma^*}\right),\\
&\overline{N}_1^2 = -\mathrm{i}\left(\alpha\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\omega} + \beta^*\frac{\partial}{\partial\alpha^*} + \omega^*\frac{\partial}{\partial\gamma^*}\right),\\
&\overline{N}_1^3 = \frac{\mathrm{i}}{2}m\alpha + 2\mathrm{i}\alpha + \mathrm{i}\left(\alpha^2\frac{\partial}{\partial\alpha} + \alpha\beta\frac{\partial}{\partial\beta} + \alpha\gamma\frac{\partial}{\partial\gamma} + \gamma\beta\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\alpha^*}\right),\\
&\overline{N}_1^4 = \frac{\mathrm{i}}{2}m\gamma + 2\mathrm{i}\gamma + \mathrm{i}\left(\alpha\gamma\frac{\partial}{\partial\alpha} + \alpha\omega\frac{\partial}{\partial\beta} + \gamma^2\frac{\partial}{\partial\gamma} + \gamma\omega\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\gamma^*}\right),\\
&\overline{N}_2^2 = -2\mathrm{i} - \mathrm{i}\left(\beta\frac{\partial}{\partial\beta} + \omega\frac{\partial}{\partial\omega} + \beta^*\frac{\partial}{\partial\beta^*} + \omega^*\frac{\partial}{\partial\omega^*}\right),\\
&\overline{N}_2^3 = \frac{\mathrm{i}}{2}m\beta + 2\mathrm{i}\beta + \mathrm{i}\left(\beta\alpha\frac{\partial}{\partial\alpha} + \beta^2\frac{\partial}{\partial\beta} + \alpha\omega\frac{\partial}{\partial\gamma} + \beta\omega\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\beta^*}\right),\\
&\overline{N}_2^4 = \frac{\mathrm{i}}{2}m\omega + 2\mathrm{i}\omega + \mathrm{i}\left(\beta\gamma\frac{\partial}{\partial\alpha} + \beta\omega\frac{\partial}{\partial\beta} + \gamma\omega\frac{\partial}{\partial\gamma} + \omega^2\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\omega^*}\right),\\
&\overline{N}_3^3 = \frac{\mathrm{i}}{2}m + 2\mathrm{i} + \mathrm{i}\left(\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + \alpha^*\frac{\partial}{\partial\alpha^*} + \beta^*\frac{\partial}{\partial\beta^*}\right),\\
&\overline{N}_3^4 = \mathrm{i}\left(\gamma\frac{\partial}{\partial\alpha} + \omega\frac{\partial}{\partial\beta} + \alpha^*\frac{\partial}{\partial\gamma^*} + \beta^*\frac{\partial}{\partial\omega^*}\right).
\end{aligned}
\right.
\tag{3.6}
$$

With (3.4)...(3.6) we obtain the following expresion (extension formulas) for the generators N_β^α:

$$N^{\alpha}_{\beta} = \rho P^{\alpha}_{\beta} + i[C_2, P^{\alpha}_{\beta}], \tag{3.7}$$

where $C_2 = \frac{1}{2} M^{\alpha}_{\beta} M^{\beta}_{\alpha}$ is the quadratic Casimir operator of $SU(4)$. Or, this is just the Gell-Mann formula [2]. Obviously, by the following contraction :

$$\lim_{\rho \to \infty} \frac{1}{\rho} N^{\alpha}_{\beta} = P^{\alpha}_{\beta}$$

we obtain a class of unitary representations of $SU(4) \times) T_{15}$. For these representations the formula (3.7) satisfies the commutation relations (2.1 b) and (2.1 c).

On the other hand, the unitary irreducible representations of $SU(4) \times) T_{15}$ can be constructed by Goebel method [1] in a $SU(4)$ basis, in which the problem of reduction to the subgroup $SU(4)$ is solved. Now, because the reduction to the subgroup $SU(4)$ is not affected by the contraction process and defining the Casimir operators of $SU(4) \times) T_{15}$ in such a way that they to be the limits of those of $SL(4, C)$ under group contraction, we can determine what is the class of unitary irreducible representations of $SU(4) \times) T_{15}$ to which the $(r = 2,\ n_1 = n_2 = 2)$ - series is contracted. When this determination is made, we can calculate the matrix elements of N^{α}_{β} in a $SU(4)$ basis, as soon as the matrix elements of M^{α}_{β} and P^{α}_{β} in this basis are known.

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FORMULE DE EXTENSIUNE PENTRU *SL* (4, *C*)

(Rezumat)

Se descrie o metodă prin care se poate efectua reducerea reprezentărilor unitare ireductibile din seria principală degenerată $(r = 2,\ n_1 = n_2 = 2)$ a grupului $SL(4, C)$ la subgrupul său maximal compact $SU(4)$. Aceasta se realizează cu ajutorul aşa numitelor formule de extensiune, care în cazul de faţă sînt formule de tip Gell-Mann. Formulele de extensiune exprimă generatorii infinitezimali ai grupului $SL(4, C)$ funcţie de generatorii grupului produs semidirect $SU(4) \times) T_{15}$, grup pentru care problema reducerii la subgrupul $SU(4)$ este complet rezolvată de Goebel [1].

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AN ENERGY CONSERVING, STABLE DISCRETIZATION OF THE HARMONIC OSCILLATOR

BY

DONALD GREENSPAN

1. Introduction

No particle in nature exhibits motion which is exactly periodic or completely free from damping. Nevertheless, in plasma dynamics [3], [4], for example, the harmonic oscillator equation

$$m\ddot{x} + \omega^2 x = 0, \tag{1.1}$$

any nonzero solution of which is both periodic and undamped, does approximate sufficiently well various aspects of the dynamical behavior under study.

In this self-contained paper, we will show that a prototype discrete approach to mech nics [1], whose aim is to simplify the amount and quality of mathematics necessary to solve nonlinear physical problems by means of computer simulation, is, indeed, energy conserving and stable when applied to harmonic motion.

2. Discrete Mechanics

In discrete mechanics the dynamical equations are difference equations and the solutions of these equations are discrete functions. In one dimension, a basic form [1] of discrete mechanics can be summarized as follows.

For $\Delta t > 0$ and $t_k = k\Delta t$, $(k = 0, 1, 2, \ldots)$, let particle P of mass m be located at x_k at time t_k. If $v_k = v(t_k)$ is the velocity of P at t_k, while $a_k = a(t_k)$ is the accelaration of P at t_k, then we will assume the relationships

$$\frac{v_{k+1} + v_k}{2} = \frac{x_{k+1} - x_k}{\Delta t}, \quad (k = 0, 1, 2, \ldots). \tag{2.1}$$

$$a_k = \frac{v_{k+1} - v_k}{\Delta t}, \quad (k = 0, 1, 2 \ldots). \tag{2.2}$$

To relate force and acceleration at each t_k, we assume a discrete Newton's equation of the general form

$$ma_k = F(t_k), \quad (k = 0, 1, 2 \ldots). \tag{2.3}$$

To determine the motion of P from given initial conditions x_0 and v_0, one proceeds recursively, where, for each k, a_k is determined from (2.3), then v_{k+1} is determined from (2.2), and finally x_{k+1} is determined from (2.1).

Existence and uniqueness of the solution of an initial value problem follows directly from the recursive structure described above. The critical computational problem, invariably, is that of instability.

Let us show first that the above form of discrete mechanics is energy conserving when applied to harmonic motion.

3. Energy Conservation

In order to study energy conservation, it is useful to define the concept of work as follows. From an initial time t_0 to a terminal time t_n, the work W done by a force F is defined by

$$W = \sum_{i=0}^{n-1} (x_{i+1} - x_i) F_i. \tag{3.1}$$

From (2.1)...(2.3), then,

$$W = m \sum_{0}^{n-1} (x_{i+1} - x_i) a_i = m \sum_{0}^{n-1} (x_{i+1} - x_i) \left(\frac{(v_{i+1} - v_i)}{\Delta t} \right) =$$

$$= m \sum_{0}^{n-1} \left(\frac{v_{i+1} + v_i}{2} \right) (v_{i+1} - v_i) = \frac{m}{2} \sum_{0}^{n-1} (v_{i+1}^2 - v_i^2) = \frac{m}{2} v_n^2 - \frac{m}{2} v_0^2.$$

If kinetic energy K_i at time t_i is defined as usual by

$$K_i = \frac{1}{2} m v_i^2,$$

then

$$W = K_n - K_0, \tag{3.2}$$

which is the classical result that the work W is the difference of the terminal and initial kinetic energies.

Note that (3.2) is valid independently of the exact form of F in (2.3). But since the concept of potential energy does depend on the precise

structure of F, in order to proceed we must select a discrete analogue of (1.1). The analogue we choose is

$$ma_k + \omega^2 \frac{x_{k+1} + x_k}{2} = 0, \quad (k = 0, 1, 2, \ldots), \tag{3.3}$$

so that

$$F_k = -\omega^2 \frac{x_{k+1} + x_k}{2}. \tag{3.4}$$

Now, consider again (3.1). Then, from (3.4),

$$W = \sum_0^{n-1} (x_{i+1} - x_i)\left(-\omega^2 \frac{x_{i+1} + x_i}{2}\right) = -\frac{\omega^2}{2} x_n^2 + \frac{\omega^2}{2} x_0^2 .$$

If the potential energy V_k at t_k is defined by

$$V_k = \frac{\omega^2}{2} x_k^2 , \tag{3.5}$$

then

$$W = -V_n + V_0. \tag{3.6}$$

Hence, from (3.2) and (3.6)

$$K_n + V_n = K_0 + V_0,$$

which, since n is arbitrary, implies that $K + V$ is invariant of time, which is indeed the law of conservation of energy.

Note that the discretization $ma_k + \omega^2 x_k = 0$ of (1.1) would not have led to the above derivation.

4. Stability

Next, let us write (2.1)...(2.3), (3.3) in the form

$$m \frac{v_{k+1} - v_k}{\Delta t} + \omega^2 \frac{x_{k+1} + x_k}{2} = 0, \tag{4.1}$$

$$\frac{v_{k+1} + v_k}{2} - \frac{x_{k+1} - x_k}{\Delta t} = 0. \tag{4.2}$$

Consider any nonzero solution of this system of the form

$$v_k = \bar{v}\lambda^k, \quad x_k = \bar{x}\lambda^k. \tag{4.3}$$

Substitution of (4.3) into (4.1) — (4.2) and simplification implies

$$\begin{cases} \dfrac{m}{\Delta t}(\lambda - 1)\bar{v} + \dfrac{\omega^2}{2}(\lambda + 1)\bar{x} = 0, \\ \dfrac{1}{2}(\lambda + 1)\bar{v} - \dfrac{1}{\Delta t}(\lambda - 1)\bar{x} = 0. \end{cases}$$

Since the determinant of the above homogeneous system must be zero, it follows that

$$\frac{m}{(\Delta t)^2}(\lambda - 1)^2 + \frac{\omega^2}{4}(\lambda + 1)^2 = 0,$$

the roots of which are

$$\lambda_1 = \frac{2\sqrt{m} + i\,\omega\Delta t}{2\sqrt{m} - i\,\omega\Delta t}, \qquad \lambda_2 = \frac{1}{\lambda_1}. \tag{4.4}$$

But from (4.4), λ_1 and λ_2 are distinct, while $|\lambda_1| = |\lambda_2| = 1$, from which stability follows.

5. Remark

Finally, let us make a remark about computation in the field of plasmas [4]. Here the well known leap-frog formulas

$$v_{k+1/2} = \frac{x_{k+1} - x_k}{\Delta t}, \qquad a_k = \frac{v_{k+1/2} - v_{k-1/2}}{\Delta t} \tag{5.1}$$

are popular (see, e. g., [4]). Unfortunately, because x and v are never known simultaneously, conservation cannot be established, and (5.1) can be applied to (2.3) only when F is independent of v. Formulas (4.1)—(4.2) or (4.1) and (see [2]) the higher order approximation

$$\frac{3}{2}a_k - \frac{1}{2}a_{k-1} = \frac{v_{k+1} - v_k}{\Delta t}$$

can be applied without such restrictions.

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O DISCRETIZARE STABILĂ A OSCILATORULUI ARMONIC, CARE ASIGURĂ CONSERVAREA ENERGIEI

(Rezumat)

În [1] s-a propus un tip de abordare discretă a mecanicii, care simplifică volumul şi calitatea mijloacelor matematice necesare pentru rezolvarea problemelor fizice neliniare prin simulare la maşina de calcul. Se aplică oscilatorului armonic tipul de discretizare propus, arătînd că el conduce la soluţie stabilă, care asigură conservarea energiei. Se indică aplicaţii în domeniul fizicii plasmei.

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LAMINAR BOUNDARY LAYERS ALONG AN INFINITE FLAT PLATE WITH OBLIQUE SUCTION

BY

S. A. S. MESSIHA

1. Introduction

Stuart [1] considered the problem of incompressible viscous fluid past a flat plate with constant suction and no heat transfer between the fluid and the plate. Reddy [2] considered the same problem as Stuart with the introduction of a first order velocity slip and temperature jump conditions at the wall. Messiha [3] considered the same problem as Stuart but with suction which varies periodically in time about a non-zero constant mean. It has been found interesting to examine the case of oblique suction which is the aim of the present work.

Let the x'-axis be along the plate, the y'-axis normal to the plate. We assume that the flow is independent of the distance parallel to the plate and that the suction velocity is of the form $v_0'\{1 + \varepsilon A e^{i\omega' t'}\}$ towards the plate in a direction making an angle α with it so that the normal velocity v' is given by

$$v' = -v_0'\{1 + \varepsilon A e^{i\omega' t'}\} \sin\alpha, \quad 0 < \alpha < \pi . \tag{1}$$

The Navier-Stokes and energy equations relevant to the above problem are

$$\frac{\partial u'}{\partial t'} - v_0'\{1 + \varepsilon A e^{i\omega' t'}\} \sin\alpha \frac{\partial u'}{\partial y'} = \frac{dU'}{dt'} + \nu \frac{\partial^2 u'}{\partial y'^2}, \tag{2}$$

$$\frac{\partial T'}{\partial t'} - v_0'\{1 + \varepsilon A e^{i\omega' t'}\} \sin\alpha \frac{\partial T'}{\partial y'} = K \frac{\partial^2 T'}{\partial y'^2} + \frac{\nu}{c}\left[\frac{\partial u'}{\partial y'}\right]^2, \tag{3}$$

where $U'(t')$ is the external flow velocity, ν — the kinematic viscosity, v_0' — a non-zero positive mean suction velocity, K — the thermal difusivity, c — the specific heat and A is a positive constant such that $\varepsilon A \leqslant 1$.

Introducing non-dimensional quantities defined by

$$(4)\quad \begin{cases} y = \dfrac{v_0' y'}{\nu}, \quad t = \dfrac{v_0'^2 t'}{4\nu}, \quad \omega = \dfrac{4\nu\omega'}{v_0'^2}, \quad u = \dfrac{u'}{v_0'}, \quad U = \dfrac{U_0'}{v_0'}, \\ \mathcal{P} = \dfrac{\nu}{K}, \quad T = \dfrac{T' - T'_\infty}{T'_W - T'_\infty}, \quad \mathcal{E} = \dfrac{v_0'^2}{c(T'_W - T)'_\infty}, \end{cases}$$

where T'_W is a constant wall temperature, T'_∞ — the temperature at the edge of the boundary layer, $\mathcal{P}$ — the Prandtl number and $\mathcal{E}$ — the Eckert number.

Hence the non-dimensional form of (2) and (3) are

$$(5)\quad \frac{\partial^2 u}{\partial y^2} + \sin\alpha(1 + \varepsilon A e^{i\omega t})\frac{\partial u}{\partial y} - \frac{1}{4}\frac{\partial u}{\partial t} = -\frac{1}{4}\frac{dU}{dt},$$

$$(6)\quad \frac{\partial^2 T}{\partial y^2} + P\sin\alpha(1 + \varepsilon A e^{i\omega t})\frac{\partial T}{\partial y} - \frac{1}{4}\mathcal{P}\frac{\partial T}{\partial t} = -\mathcal{P}\mathcal{E}\left(\frac{\partial u}{\partial y}\right)^2,$$

with the boundary conditions

$$(7)\quad \begin{cases} \text{as } y \to \infty : u \to U, \ T \to 0; \\ \text{at } y = 0 : u = -(1 + \varepsilon A e^{i\omega t})\cos\alpha, \\ \qquad T = 1 \text{ for constant wall temperature, or} \\ \qquad \dfrac{\partial T}{\partial y} = 0 \text{ for thermally insulated wall.} \end{cases}$$

2. The Velocity Field

To solve (5) and (7), let

$$(8)\quad \begin{cases} U = 1 + \varepsilon e^{i\omega t}, \\ u = 1 + \varepsilon e^{i\omega t} - u_1(y) - \varepsilon e^{i\omega t} u_2(y). \end{cases}$$

Substitute into (5), comparing harmonic terms, the solution of (5) subject to conditions (7) is easily in the form

$$(9)\quad u = 1 - (1 + \cos\alpha)e^{-y\sin\alpha} + \varepsilon e^{i\omega t}[1 - (1 - S)e^{-y\sin\alpha} - (A\cos\alpha + S)e^{-hy}],$$

or in the form

$$u = 1 - (1 + \cos\alpha)e^{-y\sin\alpha} + \varepsilon\,(M_r \cos\omega t - M_i \sin\omega t)\,, \tag{10}$$

where

$$\left\{\begin{aligned} M_r &= 1 - e^{-h_r y}\,[(1 + A\cos\alpha)\cos h_i y + i(1 - S)\sin h_i y]\,,\\ M_i &= e^{-h_r y}\,[(1 + A\cos\alpha)\sin h_i y - i(1 - S)\cos h_i y] + i(1 - S)e^{-y\sin\alpha},\\ h &= h_r + h_i = \frac{1}{2}\,[\sin\alpha + (\sin^2\alpha + i\omega)^{\frac{1}{2}}]\,,\\ S &= 1 - \frac{4iA}{\omega}\sin^2\alpha\,(1 + \cos\alpha). \end{aligned}\right. \tag{11}$$

The non-dimensional skin-friction at the wall is

$$\begin{aligned} \tau_0 &= \frac{\tau_0'}{\rho' U_0' v_0'} = \left(\frac{\partial u}{\partial y}\right)_{y=0} = \\ &= \sin\alpha\,(1 + \cos\alpha) + \varepsilon\, e^{i\omega t}\,[(1 - S)\sin\alpha + h(A\cos\alpha + S)]\,, \end{aligned} \tag{12}$$

or in the form

$$\tau_0 = \sin\alpha\,(1 + \cos\alpha) + \varepsilon\,|\,B\,|\cos(\omega t + \theta), \tag{13}$$

where

$$\left\{\begin{aligned} &B = B_r + iB_i = (1 - S)\sin\alpha + h(A\cos\alpha + S)\,,\\ &|\,B\,|^2 = B_r^2 + B_i^2 \quad \text{and} \quad \tan\theta = \frac{B_i}{B_r}\,. \end{aligned}\right. \tag{14}$$

The above expressions are valid for values of α in the range $0 < \alpha < \pi$.

In Fig. 1 the velocity profile given by (10) is plotted against y for values of $\omega = 1$, $\omega t = \pi/2$, $\varepsilon = 0.284$ and $A = 1$. It is clear that the velocity in the boundary layer increases with increasing α at all levels from the plate.

In Fig. 2 the amplitude of skin-friction fluctuations is plotted against the frequency parameter ω for $A = 0$, 1 and diferent values of α. The amplitude of skin-friction fluctuations increases with increasing α for small values of ω but decreases for large ω in case of small α at $A = 1$. For large $\alpha\,(> \pi/3)$ the amplitude decreases with increasing α for all values of ω. For $A = 0$ the amplitude increases with increasing α for all values of ω. From Fig. 3 the phase of the skin-friction fluctuations decreases with increasing $\alpha\,(0 < \alpha < \pi/2)$ for all values of ω or A.

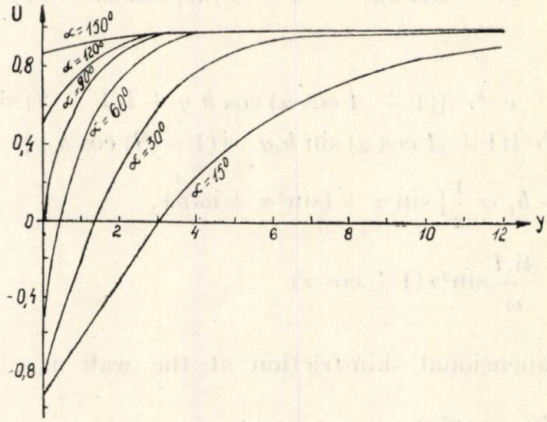

Fig. 1. — The velocity profile μ for $\omega = 1$, $\omega t = \pi/2$, $\varepsilon = 0.284$, $A = 1$.

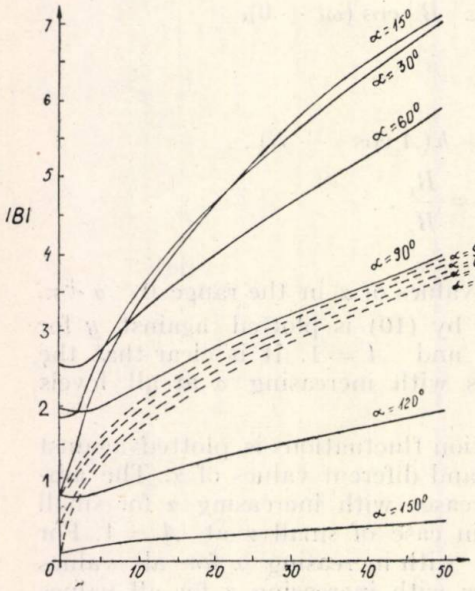

Fig. 2. — The amplitude of skin - friction fluctuations | B | against the frequency parameter ω for $A = 0$, 1 ; ——— $A = 1$; ..... $A = 0$.

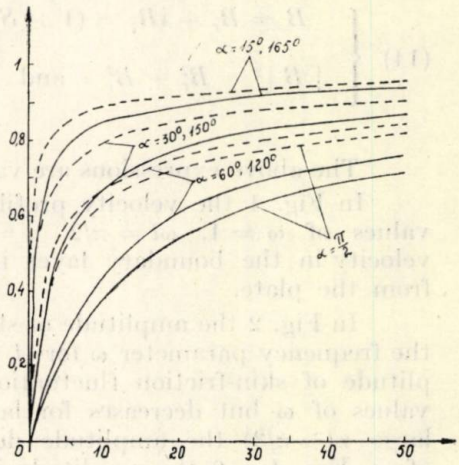

Fig. 3. — The phase of the skin - friction against ω for $A = 0$, 1 ; ——— $A = 1$, ... $A = 0$.

3. The Temperature Field

To solve (6) and (7), let

$$(15)\quad \begin{cases} T = T_0(y) + \varepsilon\, e^{i\omega t}\, T_1(y) + \varepsilon^2 e^{2i\omega t}\, T_2(y), \\ u = u_0(y) + \varepsilon\, e^{i\omega t}\, u_1(y), \end{cases}$$

where u_0 and u_1 are given from (9).

Substitute in (6), comparing harmonic terms, we get

$$(16)\quad \begin{cases} T_0'' + P \sin\alpha\, T_2' = -PEu_2'^{\,2}, \\ T_1'' + P \sin\alpha\, T_1' - \frac{1}{4} i\omega P T_1 = -AP \sin\alpha\, T_0' - 2PEu_0' u_1', \\ T_2'' + P \sin\alpha\, T_2' - \frac{1}{2} i\omega P T_2 = -AP \sin\alpha\, T_1' - PEu_2'^{\,2}, \end{cases}$$

with the boundary conditions

$$(17)\quad \begin{cases} \text{as } y \to \infty: \quad T_0,\ T_1,\ T_2 \to 0; \\ \text{at } y = 0: \ T_0 = 1,\ T_1 = T_2 = 0 \text{ for constant wall temperature, or} \\ \qquad\qquad T_0' = T_1' = T_2' = 0 \text{ for thermally insulated wall,} \end{cases}$$

where dashes denote differentiation with respect to y.

The solution of (16) satisfying the conditions at infinity are

$$(18)\quad T_0(y) = B_1 e^{-Py \sin\alpha} + \frac{PE}{2(P-2)}(1+\cos\alpha)^2\, e^{-2y \sin\alpha},$$

$$(19)\quad \begin{aligned} T_1(y) = {} & B_2 e^{-my} + \frac{4iA}{\omega} P \sin^2\alpha\, B_1 e^{-Py \sin\alpha} - \\ & - \frac{PE}{\xi} \sin^2\alpha\,(1+\cos\alpha)\left[2(1-S) - \frac{AP(1+\cos\alpha)}{P-2}\right] e^{-2y \sin\alpha} - \\ & - \frac{2PEh}{\eta} \sin\alpha\,(1+\cos\alpha)\,(A\cos\alpha + S)\, e^{-(h+\sin\alpha)y}, \end{aligned}$$

$$(20)\quad \begin{aligned} T_2(y) = {} & B_3 e^{-ny} + \frac{4iA}{\omega} B_2 m \sin\alpha\, e^{-my} - \frac{8}{\omega} A^2P^2 \sin^4\alpha\, B_1 e^{-Py \sin\alpha} - \\ & - \frac{2P^2AE}{\xi\xi_1} \sin^4\alpha\,(1+\cos\alpha)\left[2(1-S) - \frac{AP(1+\cos\alpha)}{P-2}\right] e^{-2y \sin\alpha} - \\ & - \frac{2AP^2Eh}{\eta\eta_1} \sin^2\alpha\,(h+\sin\alpha)\,(1+\cos\alpha)\,(A\cos\alpha + S)\, e^{-(h+\sin\alpha)y} - \end{aligned}$$

$$-\frac{PE}{\xi_1}\sin^2\alpha(1-S)^2\,e^{-2y\sin\alpha}+\frac{PE}{2(P-2)}(A\cos\alpha+S)^2\,e^{-2hy}-$$
$$-\frac{2PEh}{\eta_1}\sin\alpha(1-S)(A\cos\alpha+S)\,e^{-(h+\sin\alpha)y},$$

where B_1, B_2, B_3 are constants to be determined from the boundary conditions at the wall and

$$(21)\quad \begin{cases} m=\frac{1}{2}P\left[\sin\alpha+\left(\sin^2\alpha+\frac{i\omega}{P}\right)^{\frac{1}{2}}\right],\\ n=\frac{1}{2}P\left[\sin\alpha+\left(\sin^2\alpha+\frac{2i\omega}{P}\right)^{\frac{1}{2}}\right],\\ \xi=4\sin^2\alpha-2P\sin^2\alpha-\frac{1}{4}i\omega P,\\ \xi_1=4\sin^2\alpha-2P\sin^2\alpha-\frac{1}{2}i\omega P,\\ \eta=(h+\sin\alpha)^2-P\sin\alpha(h+\sin\alpha)^2-\frac{1}{4}i\omega P,\\ \eta_1=(h+\sin\alpha)^2-P\sin\alpha(h+\sin\alpha)^2-\frac{1}{2}i\omega P. \end{cases}$$

For constant wall temperature the heat transfer coefficient N at the wall for $A=0$, $P=10$, $E=0.2$ is given by

$$(22)\quad N=-\left(\frac{\partial T}{\partial y}\right)_{y=0}=10\sin\alpha\{1-0.1(1+\cos\alpha)^2\}+\varepsilon T'_{1i}-\varepsilon T'_{2r}$$

for values of the time $t=\pi/2\omega$, where

$$\begin{cases} T'_{1i}=\frac{4\sin\alpha(1+\cos\alpha)}{\eta_r^2+\eta_i^2}[(h_i\eta_r-h_r\eta_i)(\sin\alpha+h_r-m_r)+\\ \qquad +(h_i-m_i)(h_r\eta_r+h_i\eta_i)],\\ T'_{2r}=\frac{1}{8}(2h_r-n_r); \end{cases}$$

the suffices r and i denote the real and imaginary parts (of h or η or m or n) respectively.

In Fig. 4 the heat transfer coefficient N is plotted against ω for $A = 0$, $P = 10$, $E = 0.2$, $\varepsilon = 0.284$ and for different values of α. It is clear that the heat transfer increases with increasing α in the range $0 < \alpha < \pi/2$ and then decreases in the range $\pi/2 < \alpha < \pi$.

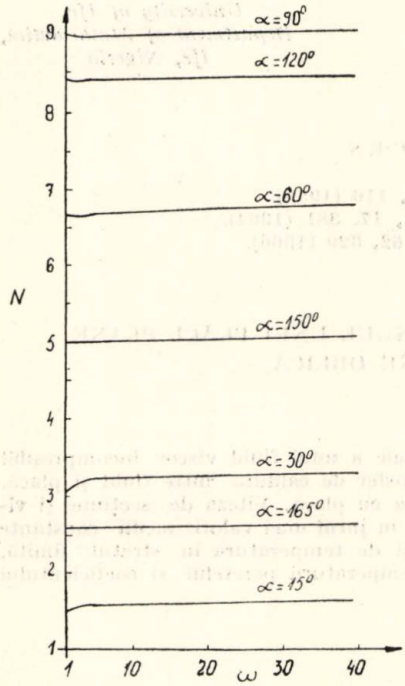

Fig. 4. — The heat transfer coefficient N against ω for $A = 0$, $E = 0.2$, $\varepsilon = 0.284$, $P = 10$, $\omega t = \pi/2$.

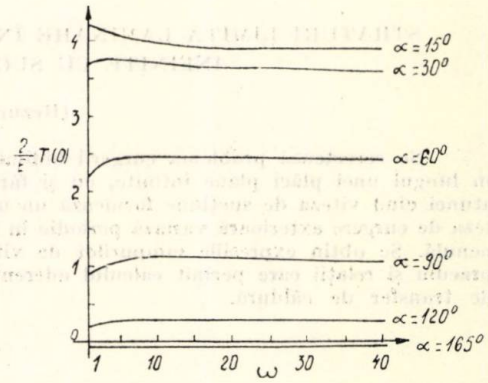

Fig. 5. — The wall temperature T against ω for $A = 0$, $P = 10$ and $\varepsilon = 0.284$.

For thermally insulated wall, the wall temperature for $A = 0$, $P = 10$ is given by

$$\frac{2}{E}\,T(0) = (1 + \cos\alpha)^2 - \varepsilon\, T_{1i} - \varepsilon^2 T_{2r} \tag{23}$$

for values of the time $t = \pi/2\omega$, where

$$\begin{cases} T_{1i} = \dfrac{40 \sin\alpha\,(1 + \cos\alpha)}{(m_r^2 + m_i^2)(\eta_r^2 + \eta_i^2)}\,[\sin\alpha + h_r - m_r)\{\eta_{1r}(h_i m_r - h_r m_i) - \eta_{1i}(h_r m_r + h_i m_i)\} + \\ \qquad + (h_i - m_i)\,\{\eta_{1r}(h_r m_r + h_i m_i) + \eta_{1i}(h_i m_r - h_r m_i)\}], \\ T_{2r} = \dfrac{5}{4}\left[1 - 2\,\dfrac{n_r h_r + n_i h_i}{n_r^2 + n_i^2}\right]. \end{cases}$$

In Fig. 5 the wall temperature is plotted against ω for $A = 0$, $P = 10$, $\varepsilon = 0.284$ and it is clear that the wall temperature decreases with increasing α for all values of ω and it becomes negative for large values of α in the range $3\pi/4 < \alpha < \pi$.

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STRATURI LIMITĂ LAMINARE ÎN LUNGUL UNEI PLĂCI PLANE INFINITE CU SUCŢIUNE OBLICĂ

(Rezumat)

Se cercetează problema curgerii bidimensionale a unui fluid vîscos incompresibil în lungul unei plăci plane infinite, cu şi fără transfer de căldură între fluid şi placă, atunci cînd viteza de sucţiune formează un unghi α cu placa. Viteza de secţiune şi viteza de curgere exterioară variază periodic în timp, în jurul unei valori medii constante nenulă. Se obţin expresiile cîmpurilor de viteză şi de temperatură în stratul limită, precum şi relaţii care permit calculul aderenţei, temperaturii peretelui şi coeficientului de transfer de căldură.

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D. C. 538.4

HYDROMAGNETIC FLOW IN A ROTATING PIPE (I)

BY

M. S. SASTRY

1. Introduction

The effects of rotation and magnetic field, acting separately, in specific problems have been well investigated. In some respects the effects are alike while in some respects the effects present a striking contrast to each other. For example, in the case of a layer of fluid neated from below, magnetic field and rotation inhibit the onset of instability. On the other hand rotation induces a component of vorticity in the direction of Ω, the angular velocity of rotation, but a magnetic field does not induce any similar component of vorticity. It is therefore of interest to study problems taking into account, the simultaneous effects of hydromagnetic and Coriolis forces. The importance of the study of such problems in many fields of geophysical and astrophysical interest need hardly be stressed [1], [2].

The present paper is devoted to a study of magnetohydrodynamic flow in a rotating pipe. The general problem of steady flow through ducts of arbitrary cross-section, rotating with an angular velocity Ω about an axis perpendicular to the axis of the duct, and subjected to the action of a uniform magnetic field, parallel to the axis of rotation is formulated as a boundary value problem.

The problem of channel flow in a transverse magnetic field in the absence of rotation has been considered by a number of workers. Mention may be made of the works of Hartmann [3], Shercliff [4], [5] Chang and Lundgren [6], [7], Sakao [8], Gold [9], [10], Chang and Yen [11] anl Chu [12]. The hydrodynamic flow in a rotating pipe has been studied by Barua [13], Vidyanidhi [14] and Nigam [15].

2. Basic Equations

The steady state hydromagnetic equations of motion and continuity in a rotating frame of reference for an incompressible fluid are :

$$\mathbf{V}\cdot \operatorname{grad} \mathbf{V} + 2\,\Omega \times \mathbf{V} = -\frac{1}{\rho}\operatorname{grad} p + \nu\,\nabla^2\mathbf{V} + \frac{1}{\rho\mu}(\operatorname{curl}\mathbf{B})\times\mathbf{B}, \tag{1}$$

$$\operatorname{div}\mathbf{V} = 0, \tag{2}$$

$$\operatorname{curl}\mathbf{B} = \mu\mathbf{J}, \tag{3}$$

$$\operatorname{curl}\mathbf{E} = 0, \tag{4}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{5}$$

together with the Ohm's law for a moving conductor

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}) \tag{6}$$

where $p = p' - \frac{1}{2}\rho\,|\,\Omega\times\mathbf{r}\,|^2$; p' denoting the fluid pressure and $\mathbf{r}$ denoting the position vector from the axis of rotation. ν, ρ, μ and σ are respectively the kinematic coefficient of viscosity, the density, the magnetic permeability and the conductivity of the fluid and Ω is the angular velocity of the system referred to a fixed inertial frame. It is to be noted that in the above equations $\mathbf{E}$ is the electric field relative to the rotating frame. If $\mathbf{E}_I$ denotes the electric field referred to a fixed inertial frame, then in a system which moves with velocity $\mathbf{V}'$ relative to the inertial frame, the electric field is given by $\mathbf{E} = \mathbf{E}_I + \mathbf{V}'\times\mathbf{B}$, where $\mathbf{B}$ is the magnetic field.

Taking curl of Eq. (3) and using (5) we have

$$\nabla\times\mathbf{J} = \frac{1}{\mu}\nabla\times(\nabla\times\mathbf{B}) = \frac{1}{\mu}[\nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B}] = -\frac{1}{\mu}\nabla^2\mathbf{B}. \tag{7}$$

Curl of Eq. (6) and Eqs. (4) and (7) yield

$$\mu\sigma\nabla\times(\mathbf{V}\times\mathbf{B}) + \nabla^2\mathbf{B} = 0. \tag{8}$$

We choose a cartesian coordinate system with z-axis along the axis of the pipe and x and y axes in the cross-sectional plane. The axis of x coincides with the axis about which the pipe rotates with uniform angular velocity. It is assumed that the wall of the pipe is of constant thickness and its permeability is approximately equal to the permeability of the fluid.

As we can see, there are three regions of interest, region *1* through which an electrically conducting fluid of electrical conductivity σ_1 flows, region *2* of electrical conductivity σ_2 comprises the wall of the pipe and the region *3* outside of the pipe of zero conductivity. At infinity, a uniform magnetic field of strength B_0 acts in the x - direction.

Since the pipe is infinitely long all physical quantities except the pressure will be independant of z. It then follows from the momentum

equation in the z - direction that $\partial p/\partial z$ is independant of z. As in the case of flow in a non-rotating pipe we shall assume that $\partial p/\partial z$ is a constant.

In the absence of rotation the flow will be one-dimensional in the sense that there will be only one velocity component. But rotation of the pipe about the x-axis produces Coriolis forces which induce flow in the y - direction. From the equation of continuity it then follows that there must be flow in the x - direction also. Thus the flow is fully three - dimensional.

We write

$$\mathbf{V} = (u,\ v,\ w), \qquad \mathbf{B} = (\bar{B}_x,\ \bar{B}_y,\ B_z). \tag{9}$$

Eqs. (2) and (5) can be indentically satisifed by introducing the functions Ψ and Φ such that

$$u = -\frac{\partial \Psi}{\partial Y}, \qquad v = \frac{\partial \Psi}{\partial X},$$

$$B_x = -\frac{\partial \Phi}{\partial Y}, \qquad B_y = \frac{\partial \Phi}{\partial X}.$$

Writing Eqs. (1) and (7) in component form in terms of Φ and Ψ, eliminating the pressure gradient by cross differentiation and substraction yields in dimensionless form,

$$\nabla^4\Psi + \mathcal{T}\frac{\partial W}{\partial X} + \mathcal{M}^2 J\begin{pmatrix}\Phi & \nabla^2\Phi \\ X & Y\end{pmatrix} - \mathcal{R}\, J\begin{pmatrix}\Psi & \nabla^2\Psi \\ X & Y\end{pmatrix} = 0, \tag{10}$$

$$\Delta^2 W + 4 + \mathcal{M}^2 J\begin{pmatrix}\Phi & B_z \\ X & Y\end{pmatrix} - \mathcal{T}\frac{\partial \Psi}{\partial X} - \mathcal{R}\, J\begin{pmatrix}\Psi & W \\ X & Y\end{pmatrix} = 0, \tag{11}$$

$$\nabla^2\Phi + \mathcal{R}_m J\begin{pmatrix}\Phi & \Psi \\ X & Y\end{pmatrix} = 0, \tag{12}$$

$$\nabla^2 B_z + \mathcal{R}_m\left[J\begin{pmatrix}\Phi & W \\ X & Y\end{pmatrix} + J\begin{pmatrix}B_z & W \\ X & Y\end{pmatrix}\right] = 0, \tag{13}$$

where

$$X = \frac{x}{a}, \qquad Y = \frac{y}{a}, \qquad U = \frac{4u}{ca^2/\nu}, \qquad V = \frac{4v}{ca^2/\nu},$$

$$W = \frac{4w}{ca^2/\nu}, \qquad B_x = \frac{\bar{B}_x}{B_0}, \qquad B_y = \frac{\bar{B}_y}{B_0}, \qquad B_z = \frac{\bar{B}_z}{B_0}.$$

a is the characteristic length, $c = -\frac{1}{\rho}\frac{\partial p}{\partial z}$ and

(14) $$\mathcal{T} = \frac{2\Omega a^2}{\nu} \quad \text{(Taylor Number)},$$

(15) $$\mathcal{M}^2 = \frac{4B_0^2}{\rho\mu ac} \quad \text{(Hartmann Number)},$$

(16) $$\mathcal{R} = \frac{ca^3}{4\nu^2} \quad \text{(Reynolds Number)},$$

(17) $$\mathcal{R}_m = \sigma\mu a U_m \quad \text{(Magnetic Reynolds Number)},$$

with $U_m = ca^2/4\nu$, are the dimensionless parameters characterising the flow.

3. Boundary Conditions

The boundary conditions for the velocity field in region *1* are the usual no slip conditions

(18) $$U = V = W = 0 \quad \text{on } C_1.$$

The boundary conditions for the magnetic field require a little consideration since three different regions are involved.

Since region *3* is non-conducting the current must be zero there. Eqs. (3) then gives curl $\mathbf{B} = 0$ or in component form

(19) $$\frac{\partial B_z}{\partial Y} = 0, \qquad \frac{\partial B_z}{\partial X} = 0$$

and

(20) $$\frac{\partial B_y}{\partial X} - \frac{\partial B_x}{\partial Y} = 0 \qquad \text{i.e.} \qquad \nabla^2\Phi = 0.$$

From Eq. (19) since B_z tends to zero at infinity, we have $B_z = 0$.

The only solution of Eq. (20) which remains bounded at infinity and tends to the applied field there is

(20 a) $$B_x = 1 \quad \text{and} \quad B_y = 0.$$

Across the boundaries C_1 and C_2 the tangential component of electric field must be continuous. Using Ohm's law the condition on the electric field can be written as

(21) $$\left[\frac{J_t}{\sigma}\right] = 0,$$

where J_t is the current in the direction of the tangent to the interface. Square bracket indicating the discontinuity in J_t.

Eq. (3) in component form gives

$$\frac{1}{\mu}\frac{\partial B_z}{\partial Y} = J_x, \qquad -\frac{1}{\mu}\frac{\partial B_z}{\partial X} = J_y.$$

This shown that B_z can be thought of as a stream function for μJ. Therefore, we have $\mu J_t = -\dfrac{\partial B_z}{\partial n}$ where n is the outward normal. Condition (21) expressed in terms of B_z becomes

$$\frac{1}{\sigma_1}\frac{\partial B_z}{\partial n_1} = \frac{1}{\sigma_2}\frac{\partial B_z}{\partial n_2}, \tag{22}$$

since $\mu_1 \approx \mu_2$.

Continuity of the magnetic induction gives further conditions as

$$ {}_1B_z = {}_2B_z, \quad {}_1\Phi = {}_2\Phi, \quad \frac{\partial({}_1\Phi)}{\partial n} = \frac{\partial({}_2\Phi)}{\partial \eta} \text{ on } C_1, \tag{23}$$

$$ {}_2B_z = 0, \quad {}_2\Phi = {}_3\Phi, \quad \frac{\partial({}_2\Phi)}{\partial n} = \frac{\partial({}_3\Phi)}{\partial n} \text{ on } C_2. \tag{24}$$

The subscript to the left of each physical quantity corresponds to the region.

Eqs. (10) to (13) subject to the above boundary conditions are to be solved for Ψ, W, B_z and Φ.

Detailed analysis of the problem of flow in a rotating circular pipe will be presented in a subsequent paper.

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CURGEREA HIDROMAGNETICĂ PRINTR-O CONDUCTĂ ÎN ROTAŢIE (I)

(Rezumat)

Se formulează problema curgerii magneto hidrodinamice printr-o conductă lungă de secţiune transversală arbitrară. Problema conduce la un sistem de patru ecuaţii cu derivate parţiale, neliniare, care în formă adimensională conţin patru parametri, anume : $\mathcal{T}$ (numărul lui Taylor, $\mathcal{N}$ (numărul lui Hartmann), $\mathcal{R}$ (numărul lui Reynolds) şi $\mathcal{R}m$ (numărul Reynolds magnetic). Urmează o analiză amănunţită a ecuaţiilor în partea a două a lucrării.

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L'INFLUENCE DE LA GÉOMETRIE DE LA PIÈGE ET DES VARIATIONS DE LA DENSITÉ SUR L'APPARITION DES PIÈGES D'ONDES MAGNÉTOACOUSTIQUES [1])

PAR

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1. Les résultats d'un travail antérieur [1] indiquent la possibilité de la formation des pièges pour les ondes magnétoacoustiques rapides dans les pièges à symétrie axiale avec „bouchons magnétiques" crées par deux courants circulaires parallèles, respectif antiparallèles.

Ces pièges se distribuent dans le voisinage des minimums de la fonction $f = U^2/r^2$. Les calculs ont été effectués pour des pièges à géométrie particulière, où le rapport $l/a = 1$ (a est le diamètre des anneaux et l est la distance entre les anneaux). Vu que le plasma est confiné dans le voisinage du plan $z = 0$, un intérêt physique présenteront seulement les minimums situés dans le voisinage de ce plan.

Lorsque le champ est produit par deux courants opposés, grâce à la continuité du champ par rapport à la variable $\xi = l/a$, pour des valeurs de l/a proches de 1 les minimums resteront dans la proximité du plan $z = 0$, c'est à dire dans l'intérieur du domaine effectivement occupé par le plasma.

Quant aux anneaux traversés par des courants parallèles, on remarque que les minimums se distribuent sur le cercle $r = 1{,}8$; $z = 0$, par conséquant ils tombent à l'extérieur du domaine occupé par le plasma.

Nous allons prouver que ce résultat n'est pas conditionné par le choix du rapport l/a et que, pour les valeurs de ξ utilisées dans des divers modèles expérimentales de pièges pour le plasma, les minimums de la fonction f sont situés de même à l'extérieur de l'espace util de la piège.

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[1]) Recherche supporté par le Comité d'État pour l'Énergie nucléaire de R. S. R.

Puisque dans ce dernier cas les minimums paraîssent toujours dans le plan $z = 0$, nous allons utiliser l'expression du potentiel vecteur A_φ dans ce plan. En développant en série l'expression du potentiel jusqu'aux termes de III-ème ordre par rapport à r, on trouve :

$$A_\varphi = Dr\{1 - Er^2 - Cr^3\}, \tag{1.1}$$

où nous avons noté :

$$D = \frac{2M}{(1+\xi^2)^{3/2}}, \quad E = \frac{3}{8}\frac{4\xi^2 - 1}{(1+\xi^2)^2}, \quad C = \frac{245}{8(1+\xi^2)^3}. \tag{1.1'}$$

Grâce à la symétrie du problème, $B_r = 0$ et

$$B_z = Dr\left\{\frac{2}{r} - 4Er - 5Cr^2\right\}. \tag{1.2}$$

Les extrêmes de la fonction $f = \frac{U^2}{r^2} = a\frac{B_z^2}{r^2}$ avec $a = \frac{1}{\sqrt{\mu_0\rho}}$ sont données par les équations :

$$\frac{d}{d\pi}\left(\frac{U}{\pi}\right) = 0, \tag{1.3}$$

$$5Cr^3 + 2Er^2 + 1 = 0. \tag{1.3'}$$

On peut aisément constater que l'équation (3) n'a pas de solutions dans le voisinage de $r = 0$. Quant à l'équation (3'), elle n'admet non plus de solution, les coéfficients de r^2 et r^3 étant positifs pour des valeurs de l/a supérieurs à 1/2, ce qui correspond à tous les modèles de pièges connus (tableau 1) [2].

Tableau 1

Installation	l m	a m	$\frac{l}{a}$
Alice (Post)	0,30... 0,50	0,15	2... 3,6
D. S. H.	10	1	10
Ogra	12	1,80	6,6
Ixion	8,60	2,40	3,58
Magnétron ionique (Joffe)	2	0,50	4

2. Les résultats exposés en [1] ont été obtenus en considérant la densité du plasma constante. Évidemment, c'est une idéalisation considérable du problème réel, idéalisation imposée par l'absence des modèles qui

puissent décrire la variation de la densité du plasma dans la piège avec la distance jusqu'à l'axe.

Cependant, pour un étude qualitatif de la formation des pièges d'ondes magnétoacoustiques à l'intérieur des pièges pour le plasma on peut supposer que la variation de la densité avec la distance est décrite par la loi

$$\rho = \rho_0 r^{-n}. \tag{2.1}$$

En l'occurence, les pièges d'ondes se distribuent dans le voisinage des minimums de l'expression

$$f = B^2 r^{n-2}. \tag{2.2}$$

Des calculs analogues à ceux effectués dans le paragraphe précédent nous conduisent aux résultats suivants :

a) Si $0 < n \leqslant 2$, aucun minimum ne paraît dans le voisinage de l'origine, quelque soit la valeur du rapport $\xi = l/a$. C'est à dire, pour une dépendance relativement faible de la densité avec r, dans l'espace util de la piège de plasma ne paraîtront point des pièges pour les ondes magnétoacoustiques, résultat conforme à [1], où on a étudié le cas $n = 0$.

b) Si $n > 2$, on trouve un minimum pour f dans l'espace util de la piège de plasma ($r < a$) si certains conditions restrictives imposées à la géometrie du piége sont satisfaites. Pour des valeurs du rapport l/a supérieurs à ξ_{cr}, la formation de pièges d'ondes est possible seulement si la variation de la densité avec r n'est pas trop forte :

$$n < n_{cr} = \frac{20C + 8E + 4}{2 - 5C - 4E} \tag{2.3}$$

(pour $\xi > \xi_{cr}$, le dénominateur de n_{cr} est positif). La valeur critique du rapport $\xi = l/a$ qui intervient plus haut est située entre 2 et 3 et satisfait l'équation algébrique

$$5C(\xi) + 4E(\xi) - 2 = 0. \tag{2.4}$$

Il en ressort que pour des valeurs pas trop grandes du rapport l/a, dans l'espace util de la piège de plasma apparaîtront des pièges, pour n'importe quelle valeur du gradient de la densité. Pour des valeurs de l/a supérieurs à une valeur critique on trouve des pièges d'ondes seulement si les gradients de densité ne sont pas trop grands.

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INFLUENŢA GEOMETRIEI CAPCANEI ŞI A VARIAŢIILOR DE DENSITATE ASUPRA APARIŢIEI CAPCANELOR DE UNDE MAGNETOACUSTICE

(Rezumat)

Se studiază influenţa geometriei capcanei de plasmă, creată de doi curenţi circulari paraleli, şi a variaţiilor de densitate asupra formării capcanelor de unde magnetoacustice. În lipsa variaţiilor de densitate, în interiorul capcanei de plasmă nu apar capcane de unde pentru nici o valoare a raportului l/a considerat. Apariţia unui gradient de densitate suficient de mare poate duce la formarea unor capcane de unde magnetoacustice.

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ON THE PROPAGATION OF RELATIVISTIC THERMO-MAGNETO-VISCOELASTIC WAVES IN A MATERIAL OF VOIGT-TYPE

BY

SIPRA SEN SARMA

1. Introduction

The studies in problems of thermo-magnetoelasticity that have emerged out of the interaction of three fields — mechanical, electromagnetic and thermal — have developed in recent years, vide, P a r i a [1], [2], [3], N o w a c k i [4], [5], K a l i s k i and N o w a c k i [6], B a k s h i [7], [8] and others. But these problems have been worked out mainly from a non-relativistic point of view. The relativistic standpoint has been considered by the present author [9] in a very recent paper.

In the present paper, an attempt will be made to extend the idea of the propagation of waves in a viscoelastic medium acted upon by electromagnetic and thermal fields. The results for all the cases have been compared with the existing literature and the salient features have been mentioned.

2. Problem and Fundamental Equations

The problem to be considered here is concerned with a viscoelastic solid acted upon by a magnetic field and a thermal field. Our object is to investigate the disturbances that arise due to the above fields from the relativistic point of view. The viscoelastic material is of Voigt-type. The fundamental equations are those which describe the fields as stated above.

The electromagnetic field is governed by the Maxwell's equations

$$
(1) \qquad \begin{cases} \operatorname{curl} \mathbf{H} = \mathbf{j} + \dfrac{\partial D}{\partial t}, & \operatorname{div} \mathbf{B} = 0, \\[2ex] \operatorname{curl} \mathbf{E} = -\dfrac{\partial B}{\partial t}, & \operatorname{div} \mathbf{D} = \rho_1, \end{cases}
$$

when

$$
\mathbf{B} = \mu_e \mathbf{H}, \qquad \mathbf{D} = \varepsilon \mathbf{E},
$$

where, $\mathbf{H}$, $\mathbf{E}$ — magnetic and electric force vectors, $\mathbf{D}$ — Maxwell's displacement vector, $\mathbf{B}$ — magnetic induction, ρ_1 — charge density, $\mathbf{j}$— current density, μ_e — magnetic permeability, ε — dielectric constant.

The generalised Ohm's law in the deformable medium is

$$
(2) \qquad \mathbf{j} = \sigma \left[\mathbf{E} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right] + \rho_1 \frac{\partial \mathbf{u}}{\partial t},
$$

where $\mathbf{u}$ is the displacement vector in the strained solid and σ is the electrical conductivity.

The stress-strain-temperature relation in a thermoviscoelastic solid of Voigt-type as given by N o w a c k i [10], is

$$
(3) \qquad \begin{cases} s_{ij} = 2\mu e_{ij} + 2\eta \dot{e}_{ij}, \quad \text{with} \\[1ex] s_{ij} = \tau_{ij} - \dfrac{1}{3}\delta_{ij}, \quad s = 3k(e - 3\alpha_t T), \\[2ex] k = \dfrac{2\mu(1+\nu)}{3(1-2\nu)}, \quad e = \varepsilon_{kk}, \\[2ex] e = \varepsilon_{ij} - \dfrac{1}{3} e\delta_{ij}, \quad \varepsilon_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i}), \end{cases}
$$

and η — Newtonian viscosity coefficient.

Again from (3), we also have,

$$
(4) \qquad \begin{aligned} \tau_{ij} = s_{ij} + \frac{1}{3} s\delta_{ij} = 2\mu(\varepsilon_{ij} - \frac{1}{3} e\delta_{ij}) + k(e - 3\alpha_t T)\delta_{ij} + \\ + 2\eta(\dot{\varepsilon}_{ij} - \frac{1}{3} \dot{e}\, \delta_{ij}), \end{aligned}
$$

where δ_{ij} is the Kronecker delta. The relativistic equations of motion in the deformable medium, as obtained by P a r i a [3] are given by,

$$
\begin{aligned}
&\rho_0\left(U_1\frac{\partial U_1}{\partial x_1}+U_2\frac{\partial U_1}{\partial x_2}+U_3\frac{\partial U_1}{\partial x_3}+\beta\frac{\partial U_1}{\partial t}\right)+\\
&+U_1\left[\left(U_1\frac{\partial \rho_0}{\partial x_1}+U_2\frac{\partial \rho_0}{\partial x_2}+U_3\frac{\partial \rho_0}{\partial x_3}+\beta\frac{\partial \rho_0}{\delta t}\right)+\right.\\
&\left.+\rho_0\left(\frac{\partial U_1}{\partial x_1}+\frac{\partial U_2}{\partial x_2}+\frac{\partial U_3}{\partial x_3}+\frac{\partial \beta}{\partial t}\right)\right]=\\
&=\frac{\partial S_{11}}{\partial x_1}+\frac{\partial S_{21}}{\partial x_2}+\frac{\partial S_{31}}{\partial x_3}+(\mathbf{j}\times B)_1+\rho_1 E_1,
\end{aligned}
\tag{5}
$$

and similar other two equations, where

$$
\left\{
\begin{aligned}
&U_i=\frac{\mathrm{d}u_i}{\mathrm{d}t},\\
&S_{ij}=S_{ji}=\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13} & 0\\ \tau_{21} & \tau_{22} & \tau_{23} & 0\\ \tau_{31} & \tau_{32} & \tau_{33} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.
\end{aligned}
\right.
\tag{6}
$$

The thermal field is described by the equation (vide, P a r i a [1])

$$
k\nabla^2 T=\rho c_v\frac{\partial T}{\partial t}+T_0 m\frac{\partial e}{\partial t},
\tag{7}
$$

neglecting the Joule heat, where

$$
m=3k\alpha_t.
\tag{8}
$$

The symbols used here have their usual meanings as explained in the references quoted. The equations from (1) to (7) constitute the set of equations in the theory of relativistic magneto-thermoviscoelasticity. They are to be solved under prescribed conditions to be stated later on.

3. Uni-directional Motion

We consider the simplest case of the propagation of disturbancess in one direction say x-direction only. The z-axis is chosen in the direction of the initial magnetic field (primary) H_3. Then the displacement **u** has the components $(u, 0, 0)$, where $u(x, t)$ and all other functions depend on x and t only.

In this case, equation (5) with the help of (4) and (6), reduces to

$$2\rho_0 \frac{\partial U}{\partial x} + \rho_0 \frac{\partial(\beta U)}{\partial t} + U\left(U\frac{\partial \rho_0}{\partial x} + \beta \frac{\partial \rho_0}{\partial t}\right) =$$

$$(8) \qquad = \left[\left(k - \frac{2}{3}\mu\right) + 2\mu\right]\frac{\partial^2 u}{\partial x^2} - 3k\alpha_t \frac{\partial T}{\partial x} + \frac{4\eta}{3}\frac{\partial^3 u}{\partial x^2 \partial t} +$$

$$+ (j_y B_z - j_z B_y) + \rho_1 E_x.$$

Substituting the value of k from equation (2) in equation (8), we have

$$2\rho_0 U \frac{\partial U}{\partial x} + \rho_0 \frac{\partial}{\partial t}(BU) + U\left(U\frac{\partial \rho_0}{\partial x} + \beta \frac{\partial \rho_0}{\partial t}\right) =$$

$$(9) \qquad = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - m\frac{\partial T}{\partial x} + \frac{4\eta}{3}\frac{\partial^3 u}{\partial x^2 \partial T} + (j_y B_z - j_z B_y) + \rho_1 E_x,$$

where

$$(9') \qquad \lambda = \frac{2\mu\nu}{1 - 2\nu}, \qquad m = 3k\alpha_t.$$

Now, from equations (1) and (2), we find (as obtained by Paria [3]),

$$(10) \qquad \begin{cases} j_z = 0, \quad j_y = -\dfrac{\partial H_z}{\partial x} - \dfrac{\partial D_y}{\partial t} = -\dfrac{\partial H_z}{\partial x} - \varepsilon\dfrac{\partial E_y}{\partial t}, \\ B_z = \mu_e H_z, \qquad D_x = \varepsilon E_x = 0 \end{cases}$$

and

$$(11) \qquad \frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} - H_3 \frac{\partial^2 u}{\partial x \partial t} - \frac{\nu_H}{c^2}\frac{\partial^2 h_z}{\partial t^2},$$

where

$$(12) \qquad H_y = h_y, \qquad H_z = H_3 + h_z,$$

h_y and h_z being small perturbed values. Again, with the help of (10), (9) reduces to,

$$U\left(U\frac{\partial \rho_0}{\partial x} + \beta\frac{\partial \rho_0}{\partial t}\right) + 2\rho_0 U\frac{\partial U}{\partial x} + \rho_0 \frac{\partial}{\partial t}(\beta U) = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} -$$

$$(13) \qquad - \mu_e H_z\left(\frac{\partial H_z}{\partial x} + \varepsilon\frac{\partial E_y}{\partial t}\right) - m\frac{\partial T}{\partial x} + \frac{4\eta}{3}\frac{\partial^3 u}{\partial x^2 \partial t}.$$

Now, putting $\rho_0 = \rho\beta$, where ρ represents the density in the non-relativistic case, and neglecting $U\dfrac{\partial U}{\partial x}$ as small and also neglecting second order of small quantities as H_3, we obtain from (13), (vide, Sen Sarma [9])'

$$\frac{\partial^2 u}{\partial t^2} = \frac{(\lambda + 2\mu)}{\rho}\frac{\partial^2 u}{\partial x^2} - \frac{\mu_e H_3}{\rho}\left(\frac{\partial h_z}{\partial x} + \varepsilon\frac{\partial E_y}{\partial t}\right) - \tag{14}$$
$$- m\frac{\partial T}{\partial x} + \frac{4\eta}{3\rho}\frac{\partial^3 u}{\partial x^2 \partial t}.$$

Also, from (1) and (2), we get, as obtained by Sen Sarma [9],

$$\varepsilon\frac{\partial E_y}{\partial t} + \sigma E_y = \frac{H_3}{\nu_H}\frac{\partial u}{\partial t} - \frac{\partial h_z}{\partial x}. \tag{15}$$

Lastly, (7) takes the form,

$$K\frac{\partial^2 T}{\partial x^2} = \rho c_v\frac{\partial T}{\partial t} + T_0 m\frac{\partial^2 u}{\partial x \partial t}. \tag{16}$$

4. Case of Finite Conductivity

Where σ is finite, we assume

$$\begin{cases} u = u' \exp[i(\gamma x - \omega t)], \\ h_z = h' \exp[i(\gamma x - \omega t)], \\ E_y = E' \exp[i(\gamma x - \omega t)], \\ T = T' \exp[i(\gamma x - \omega t)], \end{cases} \tag{17}$$

where u', h', E', T' are constants, γ is the wave velocity and ω is real

Substituting the values of u, h_z, E_y and T in the equations (14) (16), (11) and (15) and then eliminating u', h', E', T' from the resulting equations, we have the wave velocity equation, given by

$$\begin{vmatrix} \omega^2 - \frac{\lambda+2\mu}{\rho}\gamma^2 + \frac{4i\eta}{3\rho}\gamma^2\omega & -\frac{i\mu_e H_3\gamma}{\rho} & \frac{i\varepsilon\mu_e H_3\omega}{\rho} & -\frac{im\gamma}{\rho} \\ T_0 m\gamma\omega & 0 & 0 & k\gamma^2 - i\rho c_v\omega \\ -H_3\gamma\omega & \nu_H\left(\frac{\omega^2}{c^2} - \gamma^2\right) + i\omega & 0 & 0 \\ -\frac{iH_3}{\nu_H}\omega & -i\gamma & -\sigma + i\omega\varepsilon & 0 \end{vmatrix} = 0. \tag{17 a}$$

If we define the characteristic frequency,

$$\omega^* = \frac{\rho c_v C_1^2}{K} = VC_1, \quad \text{where} \quad C_1^2 = \frac{\lambda + 2\mu}{\rho}$$

and introduce dimensionless quantities

$$
(18)\qquad \begin{cases}
\chi = \dfrac{\omega}{\omega^*} = \dfrac{\omega}{VC_1}, \quad \xi = \dfrac{\gamma C_1}{\omega^*} = \dfrac{\gamma}{V}, \\[2ex]
\epsilon_T = \dfrac{m^2 T_0}{\rho^2 c_v C_1^2} = \dfrac{m^2 T_0}{\rho k V C_1}, \\[2ex]
\epsilon_H = \dfrac{\nu_H \omega^*}{c^2} = \dfrac{\nu_H V C_1}{c^2}, \\[2ex]
R_H = \dfrac{(\mu_e H_3)^2}{\rho C_1^2}.
\end{cases}
$$

The determenantal equation simplifies to

$$
(19)\qquad \begin{aligned}
&(\mathrm{i}\epsilon V C_1 \chi - \sigma)\Bigg[\{\epsilon_H (C_1^2\chi^2 - C^2\xi^2) + \mathrm{i}C_1^2\chi\} \times \\
&\times \left\{\left[\chi^2 - \xi^2\left(1 - \frac{4\mathrm{i}\eta V}{3C_1\rho}\chi\right)\right](\xi^2 - \mathrm{i}\chi) + \mathrm{i}\epsilon_T \xi^2 \chi\right\} - \\
&- \mathrm{i}R_H C_1^2 (\xi^2 - \mathrm{i}\chi)\xi^2\chi\Bigg] + \\
&+ \mathrm{i}\epsilon R_H \frac{V C_1^5 \chi^3}{\epsilon_H c^2}(\xi^2 - \mathrm{i}\chi)(\mathrm{i}\epsilon_H \chi - 1) = 0.
\end{aligned}
$$

If we confine ourselves to the viscoelastic material of Voigt-type whose frequency of wave ω is much less than ω^* i. e. $\chi \ll 1$ we can neglect the square and higher powers of χ. In that case (19) reduces to

$$
(20)\qquad A_1 \epsilon_H \xi^6 - \mathrm{i}\xi^4 \chi \left[A_1 \epsilon_H + \epsilon_H \epsilon_T + \frac{C_1^2}{c^2}(A + R_H)\right] = 0,
$$

where

$$
(20\text{ a})\qquad A_1 = \left(1 - \frac{4\mathrm{i}\eta V}{3C_1\rho}\chi\right).
$$

Solving for ξ we have

$$
\xi = \pm (1 + \mathrm{i})\left(\frac{\chi}{2}\right)^{\frac{1}{2}}\left[1 + \frac{\epsilon_T}{A_1} + \frac{C_1^2}{\epsilon_H c^2}\left(1 + \frac{R_H}{A_1}\right)\right]^{\frac{1}{2}},
$$

which reduces to

$$
(20\text{ b})\qquad \begin{aligned}
\xi = \pm (1 + \mathrm{i})\left(\frac{\chi}{2}\right)^{\frac{1}{2}}\Bigg[&1 + \epsilon_T + \frac{C_1^2}{\epsilon_H c^2}(1 + R_H) + \\
&+ \frac{4\mathrm{i}\eta V}{3C_1\rho}\left(\epsilon_T + \frac{C_1^2 R_H}{\epsilon_H c^2}\right)\chi\Bigg]^{\frac{1}{2}},
\end{aligned}
$$

neglecting higher powers of χ and assuming

(20 c) $$\frac{4\eta V}{3C_1\rho} < 1.$$

Also, equation (20 b) can be simplified to

(21) $$\xi = \pm (1 + \mathrm{i})\left[\frac{\chi}{2}\,[(1 + \mathrm{i}b\chi)]\right]^{\frac{1}{2}}\left[1 + \epsilon_T + \frac{C_1^2}{\epsilon_H c^2}(1 + R_H)\right]^{\frac{1}{2}},$$

where

(21 a) $$b = \frac{4\eta V}{3C_1\rho}\left(\epsilon_T + \frac{C_1 R_H}{\epsilon_H c^2}\right)\Big/\left[1 + \epsilon_T + \frac{C_1^2}{\epsilon_H C^2}(1 + R_H)\right],$$

by neglecting squares and higher powers of χ and assuming that,

$$\left(\epsilon_T + \frac{C_1^2 R_H}{\epsilon_H c^2}\right)\Big/\left[1 + \epsilon_T + \frac{C_1^2}{\epsilon_H c^2}(1 + R_H)\right] < 1,$$

i.e.,

(21 b) $$1 + \frac{C_1^2}{\epsilon_H c^2} > 0,$$

which is true.

It would be worthwhile to compare this result with the corresponding results for a non-relativistic case for a viscoelastic material of Voigt-type.

In the non-relativistic case, the fundamental equations are (vide Paria [1])

(22) $$\begin{cases} \operatorname{curl} \mathbf{H} = \mathbf{j}, & \operatorname{div} \mathbf{B} = 0, \\ \operatorname{curl} \mathbf{E} = -\dfrac{\partial \mathrm{B}}{\partial t}, & \mathbf{B} = \mu_e \mathbf{H}, \\ \mathbf{j} = \sigma\left[\mathbf{E} + \dfrac{\partial \mathbf{u}}{\partial t} \times \mathbf{B}\right], \end{cases}$$

(23) $$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ik}}{\partial x_k} + (\mathbf{j} \times \mathbf{B})_i$$

and

(24) $$K\nabla^2 T = \rho c_v \frac{\partial T}{\partial t} + T_0 m \frac{\partial e}{\partial t}.$$

The stress-strain-temperature relation remain the same as (4) as in the present problem. For the uni-directional motion, we get from (22) and (23) (vide Paria [1]), and using (4)

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - m \frac{\partial T}{\partial x} - \mu_e H_3 \frac{\partial h_z}{\partial x} + \frac{4\eta}{3} \frac{\partial^3 u}{\partial x^2 \partial t}. \tag{25}$$

Also, from (22) and (24) and (25), we can have (vide, P a r i a [1])

$$\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} - H_3 \frac{\partial^2 u}{\partial x \partial t} \tag{26}$$

and

$$K \frac{\partial^2 T}{\partial x^2} = \rho c_v \frac{\partial T}{\partial t} + T_0 m \frac{\partial^2 u}{\partial x \partial t}. \tag{27}$$

Now, substituting the values of u, h_z and T from (17) in (25), (26) and (27), eliminating u', h', T' from the resulting equations, we have the wave-velocity equation,

$$\begin{vmatrix} -C_1\gamma^2 + \omega^2 + \frac{4i\eta}{3\rho}\gamma^2\omega & i\left(\frac{\mu_e}{\rho}\right) H_3\gamma & i\left(\frac{m}{\rho}\right)\gamma \\ H_3\gamma\omega & \nu_H\gamma^2 - i\omega & 0 \\ T_0 m\gamma\omega & 0 & k\gamma^2 - i\rho C_v\omega \end{vmatrix} = 0.$$

Using (18) and (20 a) the above determenantal equation simplifies to

$$R_H\xi^2\chi(\chi + i\xi^2) + (\chi + i\epsilon_H\xi^2) + [(A_1\xi^2 - \chi^2)(\chi + i\xi^2) + \epsilon_T\xi^2\chi] = 0. \tag{28}$$

Neglecting χ^2 as in the former problem, (28) reduces to

$$\xi = \pm (1 + i)\left(\frac{\chi}{2}\right)^{\frac{1}{2}}\left[1 + \frac{\epsilon_T}{A_1} + \left(1 + \frac{R_H}{A_1}\right)\Big/\epsilon_H\right]^{\frac{1}{2}}.$$

Putting the value of A_1 from (20 a) and neglecting χ^2 and higher powers of χ and using the restriction (20 c), we get from the above relation,

$$\xi = \pm (1 + i)\left[\left(\frac{\chi}{2}\right)(1 + ib'\chi)\right]^{\frac{1}{2}}\left[1 + \epsilon_T + \frac{1 + R_H}{\epsilon_H}\right]^{\frac{1}{2}}, \tag{29}$$

where

$$b' = \frac{4\eta V}{3C_1\rho}\left(\epsilon_T + \frac{R_H}{\epsilon_H}\right)\Big/\left[1 + \epsilon_T + \frac{1 + R_H}{\epsilon_H}\right]. \tag{29 a}$$

Here the condition to be imposed is

$$\left(\epsilon_T + \frac{R_H}{\epsilon_H}\right) \Big/ \left[1 + \epsilon_T + \frac{1 + R_H}{\epsilon_H}\right] < 1,$$

i.e.

$$1 + \frac{1}{\epsilon_H} > 0, \tag{29 b}$$

which is true.

Lastly, in the relativistic thermo-magneto-elastic case, as in Sen Sarma [9],

$$\xi = \pm (1 + i)\left(\frac{\chi}{2}\right)^{\frac{1}{2}} \left[1 + \epsilon_T + \frac{C_1^2}{c^2 \epsilon_H}(1 + R_H)\right]^{\frac{1}{2}}. \tag{30}$$

5. Conclusion

1. Comparing (21) and (30), we find that the effect of viscosity on the relativistic thermo-magneto-elastic wave propagation, is to increase the attenuation factor; however, the thermo-magneto coupling factors do not exhibit any marked difference.

2. Again, comparing (21) with (29), together with (30), we find that for a thermo-magneto-viscoelastic wave propagation, the attenuation factor phase velocity do not appreciably change in nature, but their magnitudes are markedly different in relativistic and non-relativistic cases. The thermo-magneto-viscoelastic coupling factors in the non-relativistic case is multiplied by the term c^2/C_1^2 for a relativistic problem.

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ASUPRA PROPAGĂRII UNDELOR TERMO-MAGNETO-VISCOELASTICE RELATIVISTE ÎNTR-UN MATERIAL DE TIP VOIGT

(Rezumat)

Se extinde la cazul relativist studiul propagării undelor într-un mediu viscoelastic supus acţiunii unor cîmpuri electromagnetic şi termic. Se constată că viscozitatea măreşte factorul de antrenare în cazul propagării undelor termo-magneto-elastice relativiste, în timp ce factorii de cuplaj termo-magnetici nu prezintă diferenţe apreciabile.

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D. C. 535.215; 546.2

THE STUDY OF THE PHOTOELECTRIC EFFECT TO Te IN THIN LAYERS

BY

N. CALINICENCO and SOFIA MELINTE

1. Introduction

It is known from the specialized literature [1], [2] that it is a „p" type semiconductor, which has an activation energy ranging from 0.34 to 0.38 eV, in the intrinsic region. [3], [4] have shown that the optical properties of Te in thin layers are different from those of the metal in massive state. In thin layers, Te occurs under two allotropic states : crystaline and amorphous.

By electronic microscopy and electronic diffraction we have found that the amorphous state occurs only for very thin layers and therefore for the first layers during the deposition. This state turns into crystaline state under the air and electron actions. The fact that the amorphous state turns quickly into crystaline state, has been established as follows : the resistance of the very thin layer in vacuum has been measured and a very high resistance of 10^{14} Ω has been found ; then there has been measured the resistance at every 15 minutes, resulting that it falls, but after about an hour it remains constant. As a crystalised layer does not show this phenomenon, we can draw the conclusion that the amorphous layer, by gas adsorbtion, changes its energy state.

2. Preparation of Samples

The layers used for this study have been deposited by thermic vaporization under a vacuum of 10^{-6} torr upon a disc-shaped 15 mm diameter glass support and having rectangular shape. We have used only surface cells with the distance between electrodes from 0.8 to 1.2 cm. Platinum

under yarn or very thin sheet form has been used as electrodes. The contacts have been realized by Te thin layer depositions. Then the ohmic aspect of the contacts at room temperature and at — 60°C, the working temperature, has been cheeked. The diagram in Fig. 1 proves what we have already said, namely, that the contacts are ohmic and the resistance rises with the bowering of temperature.

The measurements made, rendered certainly evident a diminution of the Te layer resistance, under the action of the electromagnetic waves.

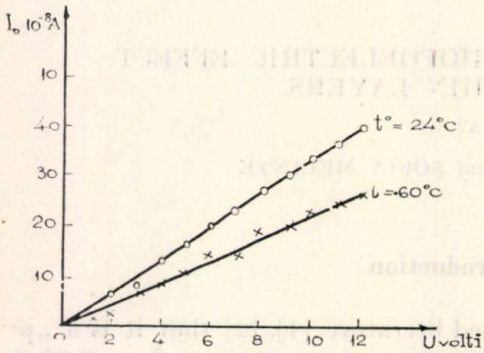

Fig. 1. — Variation in resistance with temperature for the control of the contacts ohmicity at lower temperatures.

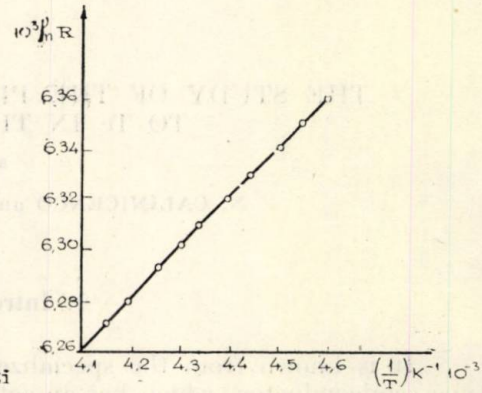

Fig. 2 — Variation of resistance as function of the absolute temperature ($\ln R = f(1/T)$).

These measurements have been made for three wavelengths : 500, 589 and 700 mμ. The measurements were made by means of a Thomson bridge, using a galvanometer whose constant was of 6.10^{-11} A/mm/m. Using a 54.5 mμ thick layer, we find that the resistance diminishes with the increase in lighting. This diminution is function of the wavelength. The measurement technics is that usual of the bridge equilibration before, and after lighting. In order to avoid the heating of contacts by lighting, a thermic filter has been used. The bright spot was focalized in the center of the Te cell. Under sample continuons lighting conditions, the absolute and relative value of the photoconductibility are

$$\Delta\sigma = \frac{\Delta R_2}{R_2 R_{abs}} \quad \text{and} \quad \Delta\sigma R_{abs} = \frac{\Delta R_2}{R_2}, \tag{1}$$

where $\Delta\sigma$ is the absolute conductibility value and $(\Delta R_2/R_2)$ % is the relative photoconductibility.

The values obtained for the same layer thickness and for different wavelengths are given in Table 1. One can see that for the same layer thickness, the absolute and relative photoconductibility grows with the increasing in frequence of the incident radiation.

Table 1

Sample thickness mμ	λ mμ	$10^4 R_2$ Ω	$10^4 \Delta R_2$ Ω	$10^4 R_{abs}$ Ω	$10^{-8} - \Delta\sigma$	$\Delta\sigma\ R_{abs}$	$\frac{\Delta R_2}{R_2}$ %
54.7	700	9.1	0.2	9.3	23	0.0213	2.13
54.7	589	8.1	0.4	9.2	48	0.0432	4.32
54.7	500	7.4	0.5	7.9	80	0.0622	6.22

As Te is a „p" type semiconductor, the photoconductibility is due to the emptiness shifting under the electromagnetic wave action.

It has been found that the curve $\log R = f(1/T)$, in the reversible field, is a linear one. Then the curve $R(T)$ is expressed by a relation of the form :

$$R = A \exp\left(\frac{\Delta E}{2KT}\right), \tag{2}$$

valid for semiconductors, where A is a constant characteristic of the semiconductor ; ΔE — activation energy of the conduction, prohibited zone length respectively, if the semiconductor is intrinsic ; K — Boltzmann constant value ; T — absolute temperature.

From (2) one can quickly get

$$\ln R = \frac{\Delta E}{2K}\frac{1}{T} + \ln A.$$

One can see that the angular coefficient of the line is $\Delta E/(2K) = \operatorname{tg}\varphi$, whence

$$\Delta E = 2K \operatorname{tg}\varphi. \tag{3}$$

Plotting on a graph $\ln R = f(1/T)$ for five samples there have been obtained the stright lines given in Fig. 2.

3. Conclusion

1. It is noted that in the field of reversible variation of electric resistance with temperature, the dependence of $\ln R = f(1/T)$ is perfectly linear, without bends. ΔE can be obtained introducing in the relation (3) the difference of logarithms for a temperature interval selected on the curve.

2. The value found by us for thin layers of Te is 0.344 eV, in full agreement with [1]. This energy corresponds to the width of the interdicted area and is due to the irregularity of the network which leads to the separation of the conductibility area from the valence area.

3. Reitz' reduced scheme given by [3] on the power areas of Te is in this way verified. The value of the load carrier activation power ΔE which we found experimentally corresponds to an intrinsec semiconductor.

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STUDIUL EFECTULUI FOTOELECTRIC LA Te ÎN STRATURI SUBŢIRI

(Rezumat)

Se pune în evidenţă faptul că Te în straturi subţiri se prezintă sub două stări alotropice: amorfă şi cristalină. Se confirmă faptul că Te în straturi subţiri se comportă ca un semiconductor deoarece rezistenţa lui creşte cu scăderea temperaturii. Măsurătorile efectuate, au pus de asemenea în evidenţă în mod cert o diminuare a rezistenţei straturilor sub acţiunea undelor electromagnetice luminoase. S-a determinat fotoconductibilitatea absolută şi relativă pentru o grosime $d = 54{,}5$ mμ, constatîndu-se o creştere a fotoconductibilităţii cu scăderea lungimii de undă.

Din graficul $\ln R = f(1/T)$ s-a determinat energia de activare, găsindu-se $\Delta E = 0{,}344$ eV pentru $d = 54{,}5$ mμ.

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D.C. 541.135

ON POLARIZATION PROCESSES IN PULSED ELECTRIC FIELDS [1])

BY

EUGEN NEAGU and EMIL LUCA

1. General Considerations

Starting from the observation that by the successive application of a high frequency electric field and of a constant one, the polarization processes are increased and the orientation of the dipoles takes more time [1]...[6] the forming of electrets in pulsed electric fields was studied in a previous work [7]. By using frequencies between 2 and 40 kc/s on organic glass samples, electrets were formed with properties comparable to those obtained by polarization in constant fields. Also, it was observed, that for a certain frequency of the pulsed field, which was called resonance frequency, the surface charge is larger and the stability is better than those of the electrets obtained in a constant field.

In this work subsequent experimental studies were performed on the polarization of polyetyleneterephtalate (PET) samples by pulsed electric fields with frequencies variyng between 10 c/s and 20 c/s.

2. Experimental Results

The rectangular electric pulses were obtained by a pulse generator and amplified by an amplifier specially designd by us to 3 000 V. The amplitude was measured by an oscilloscope type IO—4 and two voltage dividers were used, type DNE—7 up to 5 kc/s and type DNE—6 for the larger frequencies. For all frequencies the amplitude was kept constant by modifying continuously the amplifier anode voltage.

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[1]) Presented at the scientific session of the Polytechnic Institut of Jassy, July 2—4, 1970.

The experimental measurements were performed on disks of PET 25 mm in diameter and 1.7 mm thickners. The measurements were carried out at room temperature both for the pulsed electric field and the sinusoidal field. The signal amplitude was 1 850 V (coresponding to a field strength of 10 880 V/cm) and the frequency ranged from 10 to 20 000 c/s. The pulses were 2 ms in duration.

The measurement of the polarization was performed by a method similar to that used by C a m l i b e l [8], yielding the curves of Fig. 1 and Fig. 2.

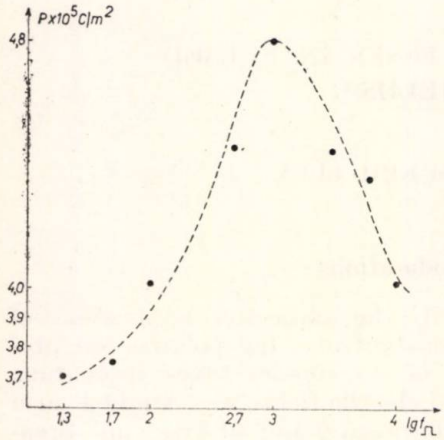

Fig. 1. — The polarization vs. frequency of pulsed field.

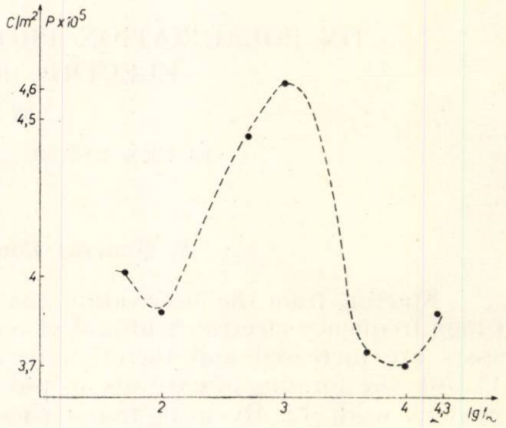

Fig. 2. — The polarization vs. frequency of sinusoidal field.

It can be seen from that the variation of the polarization with frequency is not essentially influenced by the character of the signal (pulsed or sinusoidal). Both curves have a maximum at about 1.000 c/s, indicating the presence of a resonance process at this frequency.

Besides the polarization the surface charge was also measured, both for pulsed and sinusoidal field. For this purpose the samples were kept under the influence of the field for 10 minutes. The strength of the electric field was 10 880 V/cm and the frequency ranged from 10 to 20 000 c/s.

The measurement of the surface charge was carried out by the method of electrostatic induction [9], immediately after the field was removed.

The experimental results are presented in Fig. 3 and Fig. 4. It can be seen that the surface charge varies with frequency and that the character of this variation is strongly influenced by the form of the applied signal (pulsed or sinusoidal). For the sinusoidal field it is observable that a notable surface charge is found only at smaller frequencies, up to several hundreds of c/s, and that practically no charge is found for frequencies above 1 000 c/s.

For the pulsed field the surface charge has a notable value on the whole range of frequency used.

The resonance process at 1 000 c/s with a pulsed field is also indicated by a maximum of the surface charge on the sample two faces, at this frequency.

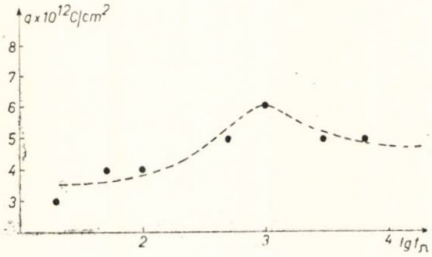

Fig 3. — Variation of the surface charge vs. frequency of pulsed field.

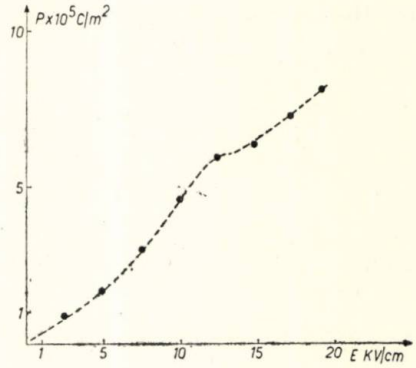

Fig. 4. — Variation of the surface charge vs. frequency of sinusoidal field.

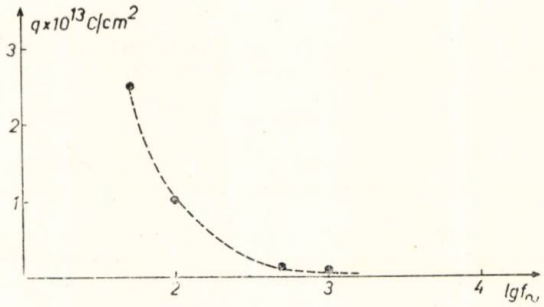

Fig. 5. — Variation of the polarization vs. strength of pulsed field.

The experimental measurements in a constant fild with the strength equal to that of the pulsed field, yielded a surface charge of 4, 5 10^{-11} C/cm^2 which is lower than the surface charge corresponding to the resonance point. The dependence of the polarization on the strength of the pulsed field was also studied on samples of PET and is presented in Fig. 5.

From the form of the curve obtained we conclude that the dependence between P and E is linear in the range studied by us with a slight change of slope for fields larger than 12000 V/cm.

3. Conclusion

The experimental investigation performed on samples of PET polarized by constant, pulsed and sinusoidal electric fields, hase shown that for a certain frequency of the pulsed field, a resonance process is present, emphasized by an increase of the polarization and by an enhancement of the surface charge on the sample face.

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ASUPRA PROCESELOR DE POLARIZARE ÎN CÎMPURI ELECTRICE PULSATORII

(Rezumat)

S-a determinat valoarea polarizaţiei şi a sarcinii superficiale în cazul unor probe de PET supuse acţiunii unor cîmpuri electrice polarizante continui, pulsatorii şi sinusoidale.

Determinările au arătat că pentru o anumită frecvenţă a cîmpului pulsator, ce corespunde unui proces de rezonanţă, polarizaţia şi sarcina supreficială de pe feţele probei au valori mai mari decît cele determinate de acţiunea unui cîmp electric continuu de aceeaşi intensitate.

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ON THE CRYSTALLINE STRUCTURE OF THE COMPOUNDS IN THE $Ca_xNi_{1-x}Fe_2O_4$ SYSTEM

BY

ANASTASIA VASILIU, M. L. CRAUS and EMIL LUCA

The crystalline structure of the compounds in the $CaO-Fe_2O_3$ system was investigated by Phillips and Muan and Bertaut et al., which determined the phase equilibrium of the $CaO - FeO - Fe_2O_3$ system [1] and calcium orthoferrite structure [2].

The aim of this note is to investigate the influence of Ni ions substitution with Ca ions on the crystalline structure of Ni ferrite, because magnetic measurements showed that over certain Ca ions content, it is possible to appear also other phases : $Ca_2Fe_2O_5$, $CaFe_2O_4$, $CaFe_4O_7$ etc.

We prepared the specimens with $x = 0.00, 0.05, 0.10, 0.15, 0.20, 0.30, 0.40, 0.50, 0.70$ by means of usual technology, by mixing, in corresponding proportions, Fe_2O_3, NiO and $CaCO_3$ powders. The samples was sintered in air at 1 175°C [3] : a set of specimens was quenched, another one was slowly cooled together with the furnace.

The disc-shaped specimens was milled up to obtaining a high degree of fineness of the powder. We made the cylinder-shaped specimens for X-ray diffraction, maximum diameter being 0.4 mm.

We used a 57.3 mm diameter camera, a pinhole by 0.5 mm, a exposure time by 8 hours and FeK α radiation (Mn filtred). The film was put in asimetric position and was read with a IZA—2 comparator ; the error made in film reading was by $\pm$ 0.02 mm and by $\pm$ 0.01 A for lattice parameter (a).

The indexing of the spinel lines was made on usual technics, the lattice parameter being obtained by the Bradley - Jay extrapolation method, which avoids the absorbtion and eccentricity corrections.

Starting from the lattice parameter found by Bradley-Jay method, we plotted, for spinel phase, the dependence :

$$(d_{hkl})_{cal} - (d_{hkl})_{obs} = \Delta d_{hkl} = f(\log \sin \vartheta).$$

which gives the error value due to absorbtion and eccentricity effect (Fig.1). In Table 1 are given the d-spacings values for the more strong lines of $CaFe_2O_4$ and $Ca_2Fe_2O_5$ and d-spacings calculated (spinel phase) and observed for the sample with $x = 0.50$ (sample nr. *8*) [1], [4].

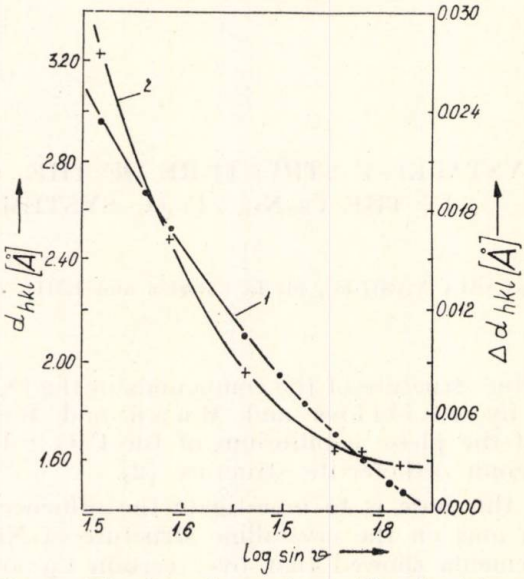

Fig. 1. — The dependences $d_{hkl} = f(\log \sin \vartheta)$ (curve *1*) and $\Delta d_{hkl} = f(\log \sin \vartheta)$ (curve *2*) for determining the absorbtion and eccentricity correction.

Table 1

Nr.	Sample nr. *8* quenched		Sample nr. *8* slowly cooled		$(hkl)_s$	d_{hkl}	
	d_{hkl} obs	d_{hkl} cal	d_{hkl} obs	d_{hkl} cal		$CaFe_2O_4$	Ca_2Fe_2O
1	2.931	2.958	2.950	2.971	220		
2	2.654	2.674*	2.662	2.675*	—	2.676	2.68
3	2.508	2.523	2.520	2.531	331	2.531	
4	2.083	2.092	2.096	2.101	400	2.115	2.07
5	1.942	1.949*	—	—	—	1.961	1.94
6	1.833	1.836*	1.834	1.836*	—	1.837	1.84
7	1.704	1.708	1.714	1.715	422		
8	1.606	1.610	1.617	1.617	333		1.62
9	1.508	1.511*	1.510	1.510*	—	1.512	1.53
10	1.476	1.479	1.485	1.485	440	1.460	1.47

Table 1 contains only d-spacings corresponding to $\vartheta \leqslant 45°$ because in this range there is the more strong lines ; the values noted with * are due to strange phases and were corrected for absorbtion and eccentricity.

In Fig. 2 is given the dependence of lattice parameter with Ca ions content. We can see that up to $x = 0.15$ the *1* and *2* curves practically are superposed, the a vs. x dependence being linear in this range.

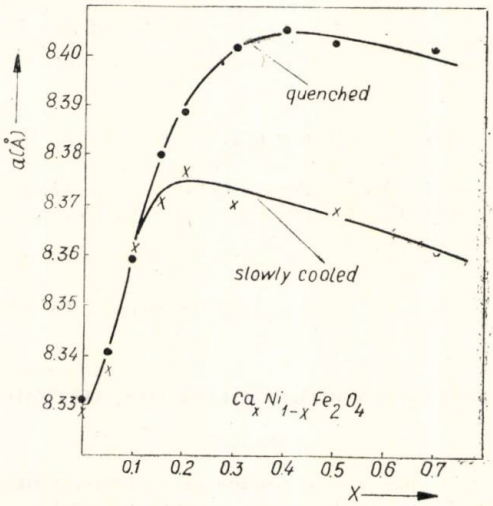

Fig. 2 — The dependence of lattice parameter a vs. Ca ions concentration x, for slowly cooled an quenched samples.

The lattice parameter for slowly cooled samples at $x=0.2°$ and for the quenched samples at $x = 0.40$, reaches a maximum ; for higher values of Ca ions content than the values corresponding to maximum, for both sets of specimens, the lattice parameters suffer a slow and quasilinear decrease.

Substituted Ni ions have a smaller volume than Ca ions and the substitution produces an increase of the unit cell, but in the same time induces microdistortions in the lattice. That fact explain the increase of lattice parameters and the gradual broadening of the lines. In our case, the broadening can be observed begining with the sample with $x = 0.05$.

From the dependence a vs. x it is expected that up to $x = 0.20$, for slowly cooled samples, and $x = 0.40$, for quenched ones, to have a solid solution from $NiFe_2O_4$ and $CaFe_2O_4$.

Over these concentrations of Ca ions, appear strange phases which (see Table 1) are $CaFe_2O_4$ and $Ca_2Fe_2O_5$, but the lines of the second phase are hidden by the lines of $CaFe_2O_4$ and spinel phase. Another compounds, that $CaFe_4O_7$, CaO, NiO, etc., cannot be observed, but it is possible that its exist in undetectable amount for our methods.

The investigations of the compounds in the $Ca_xNi_{1-x}Fe_2O_4$ system by means of X-ray diffraction show that, up to Ca ions concentrations, $x = 0.20$, for slowly cooled samples, and $x = 0.40$ for quenched ones, there is a solid solution by $NiFe_2O_4$ and $CaFe_2O_4$; over these concentrations, appear two another phases: $CaFe_2O_4$ and $Ca_2Fe_2O_5$.

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STRUCTURA CRISTALINĂ A COMPUŞILOR DIN SISTEMUL $Ca_xNi_{1-x}Fe_2O_4$

(Rezumat)

Se studiază modificările care se produc în structura feritei de nichel prin substituirea ionilor de nichel cu ioni de calciu, întrucît măsurătorile magnetice [3] indică prezenţa uneia sau mai multor faze diferite de cea spinelică.

Rezultatele studiului de difracţie, comparate cu [1], [3], [4], duc la concluzia că apar alte două faze la concentraţii de Ca peste $x = 0.20$ pentru probele răcite lent, respectiv $x = 0.40$ pentru probele călite.

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ELECTRETS BASED ON SHELLACK AND FIR-RESIN

BY

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1. General Considerations

The enlarging of the domain of practical utilization of electrets raises the problem of finding new substances that should have as good electret properties as possible — that is these substances should have a very high remanent polarization and also constant during the time. These substances should also be resistant mechanically so that they might be worked on without being deteriorated and, besides, they should be advantageous ecconomically — that is should be cheap and easily obtainable.

Taking into account the things mentioned above, we present some researches regarding the electret properties of two natural resins — shellack and fir-resin.

2. Electrets Based on Shellack

For obtaining from pure shellack some electrets under the form of disks 40 mm in diameter and 2 mm thick, the pure shellack was dissolved in alcohol, introduced in plexiglas moneds and then heated with infrared radiations till the complete evaporation of the alcohol. The samples thus obtained were then subjected to a classical treatment for obtaining thermoelectrets [1]...[3], being heated at 60°C and at the same time subjected to a continuous 10 KV/cm electric field. The results obtained regarding the value of the remanent polarization and the variation on time of the superficial electric charge are within the limits of those indicated in the literature [4].

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Because the mechanical resistance of these electrets is very low we aimed at improving it by introducing a hard binder which is not supposed to change sensibly the electret properties of the shellack.

The best results were obtained from a mixture of 25% shellack and 75% asbestos powder. This time, for preparing the electret samples the shellack dissolved in alcohol and the asbestos powder were closely mixed in proper moulds and then the mixture was heated with infrared radiations till the alcohol evaporated. The samples thus obtained, of a high durability, were worked on mechanically and disks with a diameter of 40 mm and 3 mm thick were obtained. These samples were then subjected to the normal treatment of forming thermoelectrets, that is they were kept for two hours at a constant temperature of 80°C in a 9 KV/cm electric field. The heating was then interrupted and the samples were left for one more hour, until they cooled, under the action of the electric field.

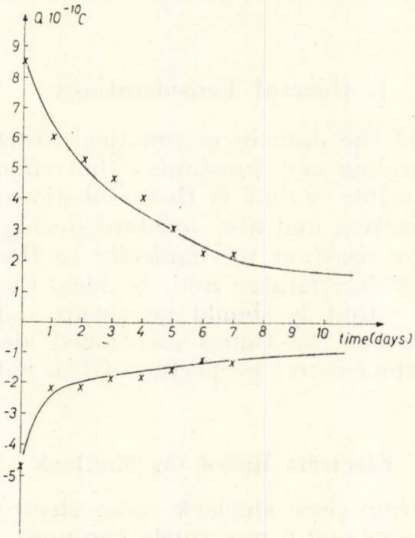

Fig. 1. — Variation on time of the superficial charge of electrets from shellack with asbestos.

In the samples thus obtained we measured the electric charge of the two faces by the method of electrostatic induction [3], [4], both immediately after the electrets were formed and later, at intervals of 24 hours.

During the measurements the electrets were kept short-circuited between tinfoil layers.

The experimental results regarding the initial value of the superficial electric charge and its variation on time are given in Fig. 1.

The values of the superficial electric charge and the way in that it varies during the time, we conclude that the samples obtained from the mixture of shellack and asbestos powder have electret properties that can be compared to those of the electrets obtained from pure shellack.

Not only the introduction of asbestos powder did not modifly the electret properties, but it improves substantially the mechanical properties of the samples.

3. Electrets Based on Fir-resin

No indications regarding the electret properties of the fir-resins have been found in the literature, a fact which required some researches in this field.

The dry fir-resin, at normal temperature, looks like an easily breakable material which, when heated at only a few degrees softens and can be modelled in any mould. Under these circumstances no useful electret can be obtained. Because of this we used again the mixture with asbestos powder. The dry resin was transformed into a very fine powder and then mixed with asbestos powder under continuous stirring for ten minutes in a closed tube glass. Then the mixture was introduced into a bronze mould and subjected to a pressure of 200 atmospheres for half an hour. At the same time the mould was heated till 45°C and then left to cool. When the sample was taken out of the mould a hard disk was obtained, 40 mm in diameter and 4 mm thick. It has been found out that the best resin-asbestos proportion is 1/3.

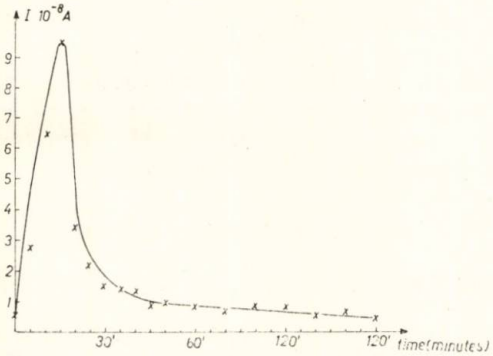

Fig. 2. — Variation of the current during the polarization of the electret from fir-resin - asbestos.

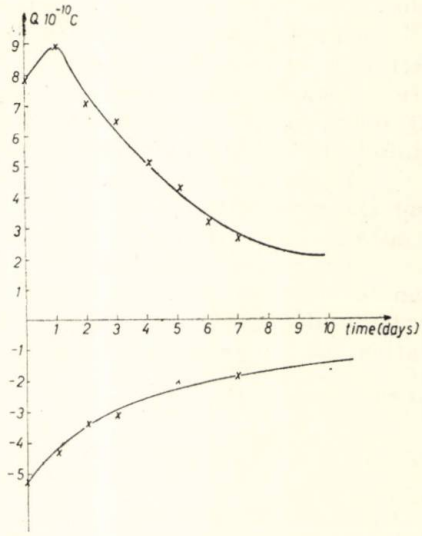

Fig. 3. — Variation on time of the superficial charge of the electrets from fir-resin - asbestos.

The disk thus obtained was then introduced between two electrodes in the furnace where the electrets are formed and it was left there under the influence of a 7 500 V/cm electric field for two hours and heated at 45°C. Having to deal with a new substance we set out, the variation curve of the current, too, by measuring it from 5 to 5 minutes (Fig. 2).

As one can see from the diagram, a maximum of the current can be obtained in 15 minutes, after which the current begins to decrease gradually until it becomes constant at a value of $0{,}5.10^{-8}$ A ; we consider that this value represents the conduction current of the material at that temperature.

The apparition of such a current maximum during the polarisation was noticed in other substances as well [2], [5] ; in our case a very sudden decrease of this current is characteristic, and this might represent the closing of the polarization process in a short time.

The heating was interrupted after two hours and the sample was left to cool in the electric field for one more hour, after which the electric charge of the faces was measured. This measurement was repeated at intervals of 24 hours, the sample being kept short-circuited between tinfoil layers. The experimental results are given in Fig. 3.

As one can see from the diagram, the heterocharge established during the polarization remains constant, its value aiming wiht time at a final constant value of about 2.25×10^{-10} C/cm^2.

From the facts mentioned above there comes out that the fir-resin can be used in good conditions for preparing some constant thermoelectrets, under the circumstances shown above, and observing the conservation rules regarding the electrets formed from organic substances.

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ELECTREŢI PE BAZĂ DE SHELLACK ŞI RĂŞINĂ DE BRAD

(Rezumat)

În scopul obţinerii unor electreţi din materiale indigene uşor de procurat, s-au realizat amestecuri din shellack sau răşină de brad şi praf de asbest.

Probele astfel formate, încălzite în cîmpuri electrice continui cuprinse între 7 500–10 000 V/cm² prezintă atît proprietăţi electrice remarcabile, comparabile cu ale celorlalte tipuri de termoelectreţi, datorită shellackului sau răşinii de brad, cît şi bune proprietăţi mecanice, datorită prafului de asbest.

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